Intro to Analysis 1 – Kalman – Fall 2011
Solutions and Comments for Assignment 4.

Section 5

These are pretty cut and dried, so I am just going to give the answers and leave it to the class to check their own papers.

5.1 (a) $(-\infty, 0)$. (b) $(-\infty, 2]$. (c) $[0, \infty)$. (d)$(-2\sqrt{2}, 2\sqrt{2})$

5.2 (a) $-\infty, 0$. (b) $-\infty, 2$. (c) $0, \infty$. (d)$-2\sqrt{2}, 2\sqrt{2}$

5.3 (h) $2, \infty$. (k) $0, \infty$. (l) $-\infty, 2$. (o) $-\infty, 0$. (t) $-\infty, 2$. (u) $0, \infty$

Section 7

For 7.3 I am again giving answers without checking the papers. I am also giving solutions to the other questions but look for comments on your papers as well.

7.3 (a) 1, (c) 0, (d) 1, (e) does not converge (dnc), (f) 1, (g) dnc, (h) dnc, (i) 0, (o) 0, (q) 0

Extra: $\sin(10^n \text{degrees})$: Note that we can reduce any angle modulo 360 without changing the sine. In particular, $\sin(1000) = \sin(-80)$ because $360|(1000 - (-80))$. Next, observe that $-80 \equiv -80 \cdot 10 \mod 360$ because $-80 - (-800) = 720$. This shows that once we reach $10^3 = 1000$, all the powers of 10 are congruent to -80 modulo 360. Therefore, for all $n \geq 3$, $\sin(10^n) = \sin(-80) = -\sin(80)$. Thus, the sequence converges and the limit is $-\sin(80)$. Curiously enough, there is an exact algebraic expression for $\sin(80)$, namely

$$\sin(80) = \sqrt[3]{\frac{\sqrt{3}+i}{16}} + \sqrt[3]{\frac{\sqrt{3}-i}{16}}$$

but that is another story.

7.4 (a) Let $x_n = (1/n)\sqrt{2}$. This is irrational for all n, for otherwise, if x_n is rational, so is $nx_n = \sqrt{2}$, which is false. The limit of (x_n) is 0, which is rational.

(b) One solution is to let r_n be the n decimal truncation of the infinite decimal for $\sqrt{2}$. That is, let (r_n) be the sequence 1.4, 1.41, 1.414, ... Each of the terms is clearly rational, and the limit is $\sqrt{2}$. This may be somewhat unsatisfying though because I have not told you how to find the decimal digits of $\sqrt{2}$. An alternative is to define $r_1 = 1$ and take

$$r_{n+1} = \frac{r_n}{2} + \frac{1}{r_n}$$

for $n \geq 1$. Again it is clear that the terms of the sequence are rational, but how do we know it converges to $\sqrt{2}$? I will leave that as an exercise but here is a hint: show by induction that $r_n^2 - 2$ decreases with n, and that it actually has to decrease to 0.

7.5 (a) Using the technique discussed in class,

$$\begin{aligned}
\sqrt{n^2+1}-n &= \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} \\
&= \frac{1}{\sqrt{n^2+1}+n} \\
&< \frac{1}{\sqrt{n^2}+n} \\
&= \frac{1}{2n}
\end{aligned}$$

so the limit is 0.

(b) Using the same approach

$$\begin{aligned}
\sqrt{n^2+n}-n &= \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \\
&= \frac{1}{\frac{1}{n}\sqrt{n^2+n}+1} \\
&= \frac{1}{\sqrt{1+1/n}+1}
\end{aligned}$$

so this time the limit is 1/2.

Extending these two problems, you might wonder if there is a general pattern for $\lim \sqrt{n^2+n^t}-n$ when $0<t<2$.

Section 8

8.1c Here is a formal proof, without any indication of how I figured it out.

Let $\epsilon>0$. Define $N=1/\epsilon$, and consider any $n>N$. By algebra,

$$\begin{aligned}
\left|\frac{2n-1}{3n+2}-\frac{2}{3}\right| &= \left|\frac{(2n-1)3-(3n+2)2}{3(3n+2)}\right| \\
&= \left|\frac{-7}{3(3n+2)}\right| \\
&= \frac{7}{3(3n+2)}.
\end{aligned}$$

Now, since $3n+2>3n$, we can apply Theorem 3.2(vii) and axiom O5 to observe that

$$\frac{7}{3(3n+2)} < \frac{7}{3(3n)}$$

$$
\begin{aligned}
&= \frac{7}{9n} \\
&= \frac{7}{9} \cdot \frac{1}{n} \\
&< \frac{1}{n}.
\end{aligned}
$$

Also, we know that $n > N = 1/\epsilon$, so again applying Theorem 3.2(vii), $1/n < \epsilon$. Combining this with the earlier result, and using transitivity, we conclude

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \epsilon.$$

The preceding steps are valid for any $n > N$, so we have fulfilled the requirements of the definition of limit, thus showing that

$$\lim \frac{2n-1}{3n+2} = \frac{2}{3}.$$

8.2e Claim: $\lim \frac{1}{n} \sin n = 0$.

Proof: Let $\epsilon > 0$. Define $N = 1/\epsilon$, and consider any $n > N$. By algebra,

$$\left| \frac{1}{n} \sin n - 0 \right| = \left| \frac{1}{n} \right| \cdot |\sin n|.$$

We know from trigonometry that $|\sin n| \leq 1$ for all n. Therefore, by order axiom O5

$$\left| \frac{1}{n} \right| \cdot |\sin n| \leq \left| \frac{1}{n} \right| = \frac{1}{n}.$$

Also, we know that $n > N = 1/\epsilon$, so applying Theorem 3.2(vii), $1/n < \epsilon$. Combining all these results and using transitivity we conclude

$$\left| \frac{1}{n} \sin n - 0 \right| < \epsilon.$$

This is valid for any $n > N$, establishing the claim.

8.5a A proof for this was distributed in class.

8.7c Let $a_n = \sin(n\pi/3)$. We are to prove that (a_n) does not converge. Arguing by contradiction, suppose that (a_n) does converge and let L be the limit. Then, taking $\epsilon = .1$ in the definition of limit, there exists some $N \in \mathbb{N}$ for which the N tail of the sequence is contained in $(L - .1, L + .1)$. In particular, since $6N + 1 > N$ and $6N + 5 > N$ we must have both a_{6N+1} and a_{6N+5} as elements of $(L - .1, L + .1)$. By direct computation, $a_{6N+1} = \sin((6N + 1)\pi/3) = \sin(2N\pi + \pi/3) = \sin(\pi/3) = \sqrt{3}/2$. Similarly, $a_{6N+5} = \sin((6N + 5)\pi/3) = \sin(2N\pi + 5\pi/3) = \sin(5\pi/3) = -\sqrt{3}/2$. But these cannot both be in an interval of length .2. Indeed, if $\sqrt{3}/2 \in (L-.1, L+.1)$, then $\sqrt{3}/2 < L+.1$ so $L > \sqrt{3}/2-.1 > 0$. At the same time, if $-\sqrt{3}/2 \in (L-.1, L+.1)$, then $-\sqrt{3}/2 > L-.1$ so $L < -\sqrt{3}/2 + .1 < 0$. As L cannot be both positive and negative, we cannot have both conditions true, and thus either $a_{6N+1} \notin (L - .1, L + .1)$ or $a_{6N+5} \notin (L - .1, L + .1)$. This is a contradiction, and so shows that (a_n) cannot converge to any limit L.