Sections 14 & 15 Homework Solutions

14.5 (a) Let $s_n = \sum_{k=1}^{n} a_k + b_k$. By commutativity and associativity of addition, we see $s_n = (\sum_{k=1}^{n} a_k) + (\sum_{k=1}^{n} b_k)$. Since $\sum a_n = A$ and $\sum b_n = B$, it is also true that $\lim_{n\to\infty} (\sum_{k=1}^{n} a_k) = A$ and $\lim_{n\to\infty} (\sum_{k=1}^{n} b_k) = B$. Thus

$\lim_{n\to\infty} s_n = A+B$, and that shows $\sum a_k + b_k = A+B$.

(b) Let $s_n = \sum_{j=1}^{n} k a_j = k\left(\sum_{j=1}^{n} a_j\right)$. By definition, $\sum a_n = A$ means $\lim_{n\to\infty}\left(\sum_{j=1}^{n} a_j\right) = A$. Thus $\lim s_n = kA$. That is $\sum k a_j = kA$.

(c) It is not reasonable to expect $\sum a_n b_n$ to equal AB. For large n, A is near $a_1 + a_2 + \cdots + a_n$ and B is near $b_1 + b_2 + \cdots + b_n$. Thus AB is close to

$$
\begin{aligned}
&(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n) \\
&= a_1b_1 + a_2b_2 + \cdots + a_nb_n \\
&\quad + a_1(b_2+\cdots+b_n) \\
&\quad + a_2(b_1+b_3+\cdots+b_n) \\
&\quad + a_3(b_1+b_2+b_4+\cdots+b_n) \\
&\quad + \text{etc},
\end{aligned}
$$

We can make a specific example using geometric series. Let $a_n = (\frac{1}{2})^n$ and $b_n = (\frac{1}{3})^n$. Then $a_nb_n = (1/6)^n$. But $\sum a_n = \frac{1}{1-1/2} = 2$, $\sum b_n = \frac{1}{1-1/3} = \frac{3}{2}$ and $\sum a_nb_n = \frac{1}{1-1/6} = \frac{6}{5}$. Since $2 \cdot \frac{3}{2} \neq 6/5$ we see $(\sum a_n)(\sum b_n) \neq (\sum a_n b_n)$.

14.6 a. Suppose $\sum |a_n|$ converges, with sum A. Also suppose (b_n) is a bounded sequence. That means there exists a real M such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Claim that $\sum |a_n b_n|$ converges. Note that, for any n, $|a_n b_n| = |a_n||b_n| \leq M|a_n|$. By exercise 14.5b, $\sum M|a_n|$ converges to MA. By theorem 14.6(i) we conclude $\sum |a_n b_n|$ converges. Then, by Corollary 14.7, $\sum a_n b_n$ converges.

14.7 Let $\sum a_n$ be a convergent series where $a_n \geq 0$ for all $n \in \mathbb{N}$. Corollary 14.5 tells us that $\lim a_n = 0$. Thus, there exists an $N \in \mathbb{N}$ such that $|a_n| < 1$ for all $n > N$. Since $a_n \geq 0$ for all n, we conclude $0 \leq a_n < 1$ for $n > N$. But then, with $p > 1$, $0 \leq a_n^p < a_n < 1$*. Thus, by theorem 14.6 (i), $\sum_{k=N+1}^{\infty} a_k^p$ converges. This shows $\sum_{k=1}^{\infty} a_k^p$ converges.

* We have not seen a proof that $0 \leq x < 1$ and $p > 1$ imply $0 \leq x^p \leq x$. For $p \in \mathbb{N}$ this follows from the order axioms by induction. For $p \in \mathbb{Q}$ we can prove this by using the fact that it holds for whole number exponents. But for $p \notin \mathbb{Q}$, we do not even have a definition of x^p (yet), let alone a proof that $x^p < x$ for $p > 1$!

14.8 If x and y are both nonnegative, then $\sqrt{xy} \le x+y$. To prove this, suppose that it is not true. Then for some nonnegative reals x and y, $\sqrt{xy} > x+y$. Then by the order axioms,

$$\sqrt{xy}\,\sqrt{xy} > \sqrt{xy}\,(x+y) > (x+y)(x+y).$$

Thus

$$xy > x^2+2xy+y^2$$

and we obtain

$$-xy > x^2+y^2.$$

This is a contradiction with $x \ge 0$ and $y \ge 0$.

Now note $|\sqrt{a_n b_n}| = \sqrt{a_n b_n} \le a_n + b_n$ for all $n \in \mathbb{N}$. Thus, ~~so~~ Theorem 14.6 (i) shows $\sum \sqrt{a_n b_n}$ converges, because $\sum a_n + b_n$ converges by exercise 14.5a.

15.6 (a) We saw in class that $\sum \frac{1}{n}$ diverges. On the other hand, $\sum \left(\frac{1}{n}\right)^2$ converges by Theorem 15.1.

(b) This is an application of exercise 14.7

(c) Given observation (b) no example is possible with nonnegative terms. So we should consider an alternating series. Let $a_n = (-1)^{n+1} \frac{1}{\sqrt{n}}$, so $\sum a_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$. This series converges by the Alternating Series Theorem. However, $\sum (a_n)^2 = \sum \frac{1}{n}$ diverges.

Extra Problems

1. Suppose (a_n) is nonincreasing and $\sum a_n$ converges. By of contradiction, suppose that a_n is not ≥ 0 for all n. Then for some particular N, $a_N < 0$. For $n \geq N$, we know $a_n \leq a_N$ because (a_n) is nonincreasing. In particular, $\lim a_n \neq 0$. Thus $\sum a_n$ does not converge. That contradicts the original assumptions, so $a_n \geq 0$ for all $n \in \mathbb{N}$ must be true.

2. First, for any fixed $k \in \mathbb{N}$ let's see that $\sum n^k r^n$ converges for all r in $(0,1)$. We use the ratio test. With $a_n = n^k r^n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^k r^{n+1}}{n^k \; r^n} = \left(\frac{n+1}{n}\right)^k r.$$

Now k is fixed, so, since $\lim \frac{n+1}{n} = 1$ we also have $\lim \left(\frac{n+1}{n}\right)^k = 1$. Thus $\lim \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, so $\limsup \left|\frac{a_{n+1}}{a_n}\right| = r < 1$. Theorem 14.8 now tells us $\sum a_n$ converges (absolutely).

Next, if $p(x)$ is a polynomial, then it is a sum of functions of the form $c_k x^k$. This shows $\sum p(n) r^n = \sum (c_0 + c_1 n + \cdots + c_k n^k) r^n$

$= c_0 \sum r^n + c_1 \sum n r^n + c_2 \sum n^2 r^n + \cdots + c_k \sum n^k r^n$

and that converges by prior results.

Extra problem 3

Let t be a real number. Consider the series $\sum a_n$ where $a_n = \frac{t^n}{n!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|t|^{n+1}}{(n+1)!} \cdot \frac{n!}{|t|^n} = \frac{|t|}{n+1}.$$

Since t is fixed, $\lim \frac{|t|}{n+1} = 0$, so $\limsup \frac{|t|}{n+1} = 0$. That is, $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ so the series

$$\sum a_n = \sum \frac{t^n}{n!}$$

is absolutely convergent by theorem 14.8