Group Work Problems

1. Suppose that (S,d) is a metric space. Prove that a set with finitely many elements is closed. (Hint: Prove that a set with one element is closed and then use the operation of set union.

2. Suppose that (S,d) is a metric space. Prove the following.
   a. If A and E are subsets of S, and if E is closed and if $A \subseteq E$, then $A^{-} \subseteq E$.

   b. If A and E are subsets of S and if $A \subseteq E$, then $A^{-} \subseteq E^{-}$.

   c. If E is a subset of S then $(E^{-})^{-} = E^{-}$.

3. Suppose that (S,d) is a metric space, and that x and y are distinct elements of S. Show that there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

4. Suppose that (S,d) is a metric space. We say that a subset D is *dense* in S if and only if $D^{-} = S$.
   a. Prove that D is dense if $D \cap U$ is nonempty for every (nonempty) open set U. [Hint. Let E be the complement of D^{-}. What can you say about E?]
   b. Prove that if D is dense, then $D \cap U$ is nonempty for every (nonempty) open set U. [Hint. Suppose that U is a nonempty open set and that $D \cap U$ is empty. Let E be the complement of U. How are D, D^{-}, and E related?]
   c. In the special case that $S = \mathbb{R}$, suppose D is a set with this property: between any two distinct reals there exists and element of D. Show that D is dense in $\mathbb{R}$ in the sense of the definition at the start of the problem.

5. Consider a fixed $\mathbf{x}$ in $\mathbb{R}^k$.
   a. Let n be any element of $\mathbb{N}$. Show that there is a point $\mathbf{r}_n$ with all rational coordinates in the ball of radius $1/n$ centered at $\mathbf{x}$. (Hint: use equation (1) page 81.)
   b. If one $\mathbf{r}_n$ is chosen for each n according to part a, show that $\lim \mathbf{r}_n = \mathbf{x}$

   [Parts a and b show that every point in $\mathbb{R}^k$ is the limit of a sequence of points with all rational coordinates.

6. Prove that the book's definition of boundary point is the same as the one I gave in class. That is, assume that (S,d) is a metric space, and that E is a subset of S. Prove that s is an element of $E^{-} \setminus E^{\circ}$ if and only if for every $r > 0$ the set $B_r(s)$ (which equals $\{x \in S : d(x,s) < r\}$) intersects with both E and with E^c. (Here, by "intersects with" I mean that the intersection is nonempty.)

7. Prove that a point is in the boundary of a set E if and only if it belongs to both the closure of E and to the closure of the complement of E.