Calculus 3
Spring 04
Exam 2 Review Sheet

The second exam will take place Friday, March 26, 2004, covering Chapter 12, sections 1 - 5 and 7 - 10, plus Chapter 13, sections 1-3. There will be a review session, most likely on Wednesday or Thursday evening. The day and time will be determined later, and posted on our webpage.

This review sheet summarizes the information you should know for the exam. A set of sample probs from Chapter 12 will be handed out separately. In studying, please also review the assigned problems from each section of the text. I have also added a few additional sample problems in the summary below.

Section 12.1

Concepts & Definitions

Real valued functions, domain, range, independent and dependent variables, interior, boundary, open, closed, bounded, unbounded, graph of a function, level curves or surfaces of a function

Procedures

Sketch graphs and level sets of functions, interpret graphs and level sets.

Section 12.2

Concepts & Definitions

Limit of function of 2 or 3 variables (in terms of mapping diagrams and the ability to define an appropriate radius in the domain within which points meet a given closeness-to-target requirement in the range), continuity, examples of functions which are not continuous at a point, two path test

Theorems

Properties of limits, *usual* functions of calculus are continuous everywhere they are defined, that is, except for instances of division by 0, negatives in squareroots, 0 or negatives in logs, etc. (This last theorem is not explicitly stated in the book, but is necessary to verify the applicability of the mixed partial theorem in the next section).

Procedures

Compute limits using limit rules, determine whether limits exist, determine whether functions are continuous, give examples or explain examples showing discontinuity

Additional Sample Problems

1. Let $f(x, y) = x^2 + y^2$. We know the limit as (x, y) approaches $(0, 0)$ for this function equals 0. State what this means using the $\epsilon - -\delta$ definition of limit. Then find an appropriate numerical value of δ to go with an ϵ of .01. Show with a diagram and with algebra why your δ is valid.
2. For this problem, use the function $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$, which is defined at every point except (0,0).
   a. If you only consider points on the x axis, and if you allow $(x, 0)$ to approach $(0, 0)$ along the x axis, then $f(x, y)$ has a limit. Show that this is true, and compute the limit.
   b. If you only consider points on the y axis, and if you allow $(0, y)$ to approach $(0, 0)$ along the y axis, then $f(x, y)$ has a limit. Show that this is true, and compute the limit.
   c. Based on parts a and b, does the limit of f as (x, y) approaches $(0, 0)$ exist? Explain.
   d. Extra credit: Let $\delta = 0.1$. Let ϵ be any positive number. Consider the circle of radius ϵ centered at the origin. Within that circle, find two points, say A and B, so that $f(A)$ and $f(B)$ are at least .2 units apart. Explain why this shows that f has no limit as (x, y) approaches $(0, 0)$

Section 12.3

Concepts & Definitions

Partial derivatives defined using limits, explained in terms of *slices*, ∂f and f_x notations, second and higher partials, mixed partials,

Theorems

Equality of mixed partials

Procedures

Find partial derivatives, use notation correctly, interpret values of partial derivatives

Additional Sample Problems

1. Questions 8 and 10 at the bottom of page 993
2. Consider the polar coordinates expression $r^2(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$, which is clearly continuous for the entire xy plane. We can express this function in terms of variables x and y with the equation $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$, which is valid for all points other than $(0,0)$. By appealing to the polar coordinates version we can see that $f(0,0)$ is equal to 0. If you use a computer graphics program to graph $z = f(x,y)$ you will see that it is a nice smooth surface. However, $f_{xy}(0,0)$ and $f_{yx}(0,0)$ are not equal.
   a. Find the formula for $f(x,y)$ at points on the x axis, and use it to show that $f_x(0,0) = 0$. Similarly, use the formula for f on the y axis to show that $f_y(0,0) = 0$.
   b. Find the formulas for $f_x(x,y)$ and $f_y(x,y)$ (for $(x,y) \neq (0,0)$) by taking partial derivatives of the formula for f.
   c. Show $f_{xy}(0,0) = -1$ as follows. Consider the formula for $f_x(x,y)$ at points of the y axis, and thinking of this just as a function of y, find the slope at $y = 0$.
   d. Use a similar approach to show $f_{yx}(0,0) = 1$
   e. Why are the mixed partials not equal? (Hint: look at $f_{xx}(x,y)$ along two lines: ($x = y$ and $x = -y$).

Section 12.4

Concepts & Definitions

Linear approximation or linearization to a function $f(x,y)$ at a specific point, differentiability (defined in terms of error in linear approximation) tangent plane to a point of the *graph* of f, normal vector to tangent plane and to graph of f, total differential

Theorems

Continuous partials near a point imply error in linear approximation can be put in the form $\epsilon_1\Delta x + \epsilon_2\Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ (Theorem 3); continuous partials in a region imply differentiability in the region; differentiability implies continuity; functions defined by formulas in calculus are differentiable except at *bad* points (This is not stated explicitly in the book, but follows from what we know about computing partial derivatives.)

Procedures

Linearize functions; use linear approximations to estimate function values; estimate error associated with linear approximation; give example of a function with partial derivatives at a point but which is not differentiable at that point

Additional Sample Problem

1. Relate the concept of differentiability for $f(x,y)$ at a specific point to the graph of f and the concept of a tangent plane. Describe what a function which is not differentiable at a point (but is continuous there) might look like.

Section 12.5

Theorems

Chain rule in various forms

Procedures

Compute derivatives both with and without using the chain rule, implicit differentiation

Additional Sample Problems

1. Consider a surface in space given by an equation of the form $f(x, y, z) = 0$, and suppose that $(1, 2, 3)$ is one point on that surface. Furthermore, suppose that $\mathbf{r}(t)$ is any space curve which lies on the surface, and with $\mathbf{r}(0) = (1, 2, 3)$. Use the chain rule to show that the tangent vector for this curve at the point $(1, 2, 3)$ is perpendicular to $\nabla f(1, 2, 3)$. [Hint: the chain rule is about derivatives when you insert one function into another. In this problem, what function can be inserted into another, and what will be the result?]

2. If $f(x, y, z) = x^2 - yz$, then we can think of f in terms of spherical coordinates, and so define partial derivatives f_ρ, f_θ, and f_ϕ. Show how to compute each of these using the chain rule.

Section 12.7

Concepts & Definitions

Gradients of functions, ∇f notation, gradient vector field of a function, chain rule using gradients, directional derivatives (defined as a rate of change of f in the direction of a unit vector $\mathbf{d}$ in the domain of f).

Theorems

Directional derivative computed as $\nabla f \cdot \mathbf{u}$; directional derivative is maximum when $\mathbf{u}$ points in the direction of ∇f, and is a minimum when $\mathbf{u}$ points in the direction of $-\nabla f$; directional derivative has a maximum value of $\|\nabla f\|$, ∇f is perpendicular to level curves, surfaces, or sets of f

Procedures

Compute gradients; compute directional derivatives; apply directional derivatives to answer question about directions of greatest increase of a function; interpret directional derivatives as slopes or as rates of change; find normal lines; normal vectors to curves and surfaces; find tangent plane to a surface

Section 12.8

Concepts & Definitions

Local maximum, local minimum, critical point, saddle point, global or absolute maximum and minimum, maxima and minima that occur at interior points or at boundary points.

Theorems

Partials equal 0 at a local max or min; second derivative test; continuous function on a closed bounded region assumes an absolute max and min.

Procedures

Find local max/min points, use first and second derivative tests, give an example where partials equal 0 but the point is neither a local max nor a local min, find global max or min over a region considering both the interior and the boundary

Additional Sample Problem

1. Let $T(x, y) = 3x^2 + y^2$ be the temperature at the point (x, y). Find the hottest point on or inside the circle of radius 1, centered at the origin.

2. For the preceding problem, give a written explanation of the rationale for your method. That is, explain why your method is certain to find the correct solution.

Section 12.9

Concepts & Definitions

Constrained maxima and minima

Theorems

Max or min for a function restricted to a curve in the domain occurs at a point where the gradient is perpendicular to the curve (in space - Theorem 9, and in the plane - Corollary to theorem 9); Lagrange Multiplier method. Note that this method only gives you *candidate* points for a max or min. The theorem says that if a max or a min exists, and if it occurs at a point where ∇g is not zero, then at the solution point $\nabla f = \lambda \nabla g$. If you find all the points that satisfy this equation, and if a max or a min exists, it must be among the points you found. However, not every problem has a solution that can be found using Lagrange multipliers. It is possible that a stated problem has no solution; or that it has a solution where $\nabla g = 0$, and in either of these cases the method of Lagrange multipliers can produce apparent solutions that are not solutions at all. So, the moral of the story is: use Lagrange multipliers to locate a max or a min but use some other analysis to convince yourself that what you have found is really the point you seek. For example, look at these two problems:

1. Find the point of the curve $x^2 + 3xy - 4y^2 = 4$ furthest from the origin.

2. Find the point of the curve $x^2 + 3xy + 4y^2 = 4$ furthest from the origin.

In each case it is possible to find solutions to the Lagrange multipliers equations, and that gives the solution to one of the problems, but not to the other. So the Lagrange multipliers technique itself cannot tell you if a max/min problem has a solution. After you have decided by some other method that a max/min problem does have a solution, Lagrange multipliers can help you determine where that solution is.

Procedures

Method of Lagrange Multipliers for constrained optimization problems.

Section 12.10

Concepts & Definitions

Taylor approximations for functions of two or more variables

Theorems

Second derivative test (now proved using quadratic approximation to a function at a point where the first partials are 0); Taylor's formulas on page 992

Procedures

Find the quadratic and cubic approximations to a function at the origin; estimate the error from these approximations

Section 13.1

Concepts & Definitions

Double integral as limit of sums of rectangular solids; interpretation as volume under *The Graph* of a function over a region in the plane; computation as iterated integral (see page 1003); limits of integration as directions for painting a region with horizontal or vertical brush strokes.

Theorems

Fubini's theorem: for a continuous function f, double integral can be done in either order with appropriate limits of integration.

Procedures

Given a region find limits of integration; given the limits of integration find the region; given a double integral in one order cange it to an integration in the opposite order; compute double integrals.

Section 13.2

Concepts & Definitions

Double integral as a method to find areas of plane regions; to find volumes of solids; to find average value of a function of 2 variables; reductionism and finding center of mass or moment of inertia. **NOTE: do not memorize formulas in 13.1 page 1014. If these are needed on the exam you will be provided them. You are not responsible for radius of gyration.**

Procedures

Finding areas, volumes, average value, center of mass, etc. using double integrals.

Section 13.3

Concepts & Definitions

Polar coordinates regions; element of area $dxdy = rdrd\theta$; painting regions with radial lines and circular arcs.

Procedures

Computing polar coordinate integrals; changing rectangular integrals to polar coordinates and vice versa.