Teaching Linear Algebra: What are the Questions?

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Who knows how to teach linear algebra? With this question, Charles Johnson began his banquet address at the 1995 ATLAST [1] workshop in Williamsburg, Virginia. Hardly a soul raised a hand, although all the participants were seasoned teachers of linear algebra. Johnson went on to declare his own ignorance regarding the *right* way to teach linear algebra, in spite of studying and teaching the subject for years. This was a sobering declaration from someone with his credentials. He is the co-author of standard references on matrix theory [14], [15], an eminent researcher in matrix analysis, and he has also been a national leader in efforts to improve the way linear algebra is taught [4].

How easy it is to slip into a routine when you have taught a course a few times. The material to be covered, the logistics of the course and grades, the pace to set, all become familiar. The instructor is in command of the situation, with little hesitation about what must be done each day. On that level, most college mathematics faculty know how to teach linear algebra.

But as Johnson's remarks suggest, there are deeper issues that deserve some thought. Introductory linear algebra courses today serve a large and very diverse audience. How much emphasis should be placed on proofs? How abstract should the setting be? What contribution, if any, can be made by technology? How do people learn mathematics, and what teaching methods are most effective?

Concern about these questions has fueled a healthy interest in *linear algebra reform* during the past decade, in much the same spirit as calculus reform. After all the debate and reflection, after articles, special issues, and books, after summer workshops and special sessions at the national meetings, after careful study by panels of experts, what can now be said about how to teach linear algebra?

That question was the focus of the Undergraduate Faculty Program at the Park City Mathematics

[1] ATLAST is an acronym for *Augment the Teaching of Linear Algebra through the use of Software Tools*. Funded by the NSF, the ATLAST Project presented a series of workshops in 1992-97 to acquaint linear algebra teachers with uses of instructional technology. Some of the materials developed at the workshops have been published in the volume [20]. There is a website for the project at http://www.umassd.edu/SpecialPrograms/Atlast/welcome.html.

Institute July 12-August 1, 1998. Each summer, the Park City Mathematics Institute (PCMI) brings together a broad spectrum of mathematics researchers, teachers, and students in a collection of coordinated programs organized around a central theme. In 1998 there were programs for mathematics researchers, graduate students, undergraduate students, undergraduate faculty, mathematics education researchers, and K-12 teachers, with a central theme of Representation Theory of Lie Groups. Each subgroup had a rich program, and participants from each had many opportunities to interact in structured and unstructured ways with members of the other groups.

PCMI is a unique event, a microcosm of the world of mathematics. It is valuable because all levels of mathematics depend on each other, and teachers at every level share a critical interest in improving mathematics education. This common concern, and the obvious need for coordination between elementary, secondary, college, and graduate mathematics curricula, ought to provide a natural context for collaboration between teachers at different levels. And yet, this kind of collaboration is rare. The opposite is true at PCMI. It provides a setting in which the study of topics narrowly focused within a single level and the consideration of issues that span all levels exist side by side.

In summer 1998 there were 13 participants in the Undergraduate Faculty Program (UFP), where for three weeks teaching linear algebra was the focus of our readings, presentations, and discussions. Members of the UFP group also participated in the other PCMI programs: Mathematics Education Research, Secondary Teaching, the courses on Lie Theory and Continuous Geometry taught by Roger Howe and Bill Barker for the Undergraduate Student Program, and the Research Lectures.

The UFP group considered goals and content, audiences and teaching approaches, texts and technology. As a result, did we master the art of teaching linear algebra? No. In fact, we quickly realized it is an oversimplification to think that there is one right way. Although we all often speak of *the first linear algebra course* as if it were the same everywhere, the truth is far richer and more diverse. Even our small group described linear algebra courses that reflect a broad variety of goals, audiences, and approaches.

We did not come away from PCMI with prescriptions. But we did obtain a deeper understanding of the questions any linear algebra teacher should consider when developing his or her own best way to approach the subject. The purpose of this article is to share some of these questions, along with resources and opinions we found thought provoking. This will be a summary; several detailed reports on different aspects of our program, and related resources, are available on-line at the UFP website: http://knox.knox.edu:5718/~aleahy/pcmi/.

In writing this account, we have benefited from discussions with the other UFP participants, but the viewpoints here, as well as any errors or inaccuracies, are our own.

The Core Curriculum

In 1990, the Linear Algebra Curriculum Study Group produced a recommended core curriculum for a first linear algebra course. The Study Group was a panel of linear algebra teachers from a diverse selection of colleges and universities. They heard from a variety of experts who use linear algebra, and they developed their report after much discussion about who needs linear algebra and why, what is reasonable to teach in a first course, and how computers might affect these things. Although their report was not officially published until 1993 [4], its impact was felt very soon via special sessions at national meetings.

Much has happened since 1990. The content of many textbooks reflects the LACSG recommendations, and of course computer software and hardware have become more powerful and less expensive. The UFP participants considered the idea of trying to update this recommended curriculum, in light of how it is actually being used today. However, in the UFP there were as many ways to construct a first linear algebra course as there were participants. Different institutions have quite different audiences for linear algebra, and they expect the first linear algebra course to play different roles in the larger curriculum. Some have a large population of engineering students, with an emphasis on physical science applications, notably differential equations and signal processing. At other schools a large proportion of the linear algebra students are mathematics majors, many of whom intend to teach at the secondary level. Some colleges teach linear algebra as a subset of a differential equations course.

Teachers' attitudes differ. Some see the first linear algebra course as part of the calculus sequence, with an emphasis on procedures and concepts relevant to n dimensional real space. Others see linear algebra as a first college course emphasizing the construction and understanding of proofs.

For each different variation, someone in the UFP group had a well thought out and carefully considered rationale, explaining why this formulation of the course makes sense in its particular context. It was soon clear that no single model curriculum would serve the needs of all these different approaches, and we abandoned any hope of either updating the LACSG recommendations or formulating our own.

Rather, we began to discuss the fundamental question: What is the first linear algebra course supposed to accomplish, given your particular setting? We realized that a first step toward answering this is to decide what curriculum model makes sense at your school. That means your faculty must pay attention to what client disciplines are represented among their linear algebra students, and to discuss with faculty in those other disciplines what topics they hope their students will learn. Few of us had ever engaged in such dialogue.

We found that the thoughtful reflections of others about curriculum can be helpful. The recommended core curriculum is a valuable framework for discussion. Other references that make recommendations regarding content are [7, 9], although these address much broader pedagogical concerns as well. A very narrowly focused article by Axler [1] argues for the exclusion of determinants from linear algebra, and presents an elegant mathematical development of eigenvalues and eigenvectors that makes no use of determinants. A related article by McWorter and Meyers [22] provides simple algorithms for hand computation of eigenvalues and eigenvectors, again without determinants. These articles are interesting if only because the viewpoints they take are so atypical.

Textbooks

There are many textbooks to choose among, and we made no attempt in the UFP to survey them. However, we did discuss some alternatives in the context of the core curriculum, and we looked at a small number of different kinds of books in some detail. For some of these, reviews by UFP participants can be found at the UFP website.

Connections with Other Mathematics Courses

There are obvious points of contact between linear algebra, calculus, and differential equations, as well as other areas such as statistics and modern algebra. The question of how to handle these connections is worthy of consideration by linear algebra teachers. However, the prerequisites for linear algebra definitely affect what topics students can learn and how they can use them. If multivariable calculus is a prerequisite, then ideas about two and three dimensions that were presented in the calculus course can be more easily generalized in linear algebra. If instead students are required to take linear algebra first, then it can be used freely in multivariable calculus in the discussion of topics such as total derivative and change of variable. At many schools, neither of these courses is a prerequisite for the other. In that case, ideas like geometry in 3 space must be developed independently as needed in each course.

At a higher level, one might consider developing the entire linear algebra course as part of some larger context that provides motivation and meaning. Differential equations is one such context. In light of the 1998 PCMI focal topic, we would be remiss not to suggest representation theory of Lie groups as another possibility. Although this would definitely not be suitable for some audiences, it is a fascinating area of mathematics, and one with connections to many areas of active research. At the very least, matrix groups provide some interesting examples that can be put to good use in linear algebra. And for honors students or for an upper division course for mathematics majors, Lie groups offer a setting in which a good part of linear algebra finds a natural home. Some of these ideas are being developed in an elementary treatment of Lie theory, parts of which were presented in the undergraduate program at PCMI by Roger Howe. An overview of Lie theory, and its connections with undergraduate topics, is provided by Howe's paper [16].

History

The historical development of any subject can often provide insights and interesting background for teachers. Although there was some discussion of historical sources (for example [19]), this was not a heavily emphasized topic during the UFP. But this is an area that can enrich courses and deserves consideration.

Connections with Secondary Mathematics

Guershon Harel ([9]) has pointed out that calculus rests on a foundation of several years of background study at the secondary level, while linear algebra demands mastery of a number of critical ideas with little or no prior foundation. He goes on to propose that students be exposed to linear algebra at the secondary level, so that in college they have a suitable basis for abstraction and continued study.

We liked this idea and the PCMI program provided a perfect setting within which to consider it. We brainstormed about what linear algebra ideas and experiences we wish were introduced at the secondary level. Our list included matrix operations and calculation, interpretations of vectors, linearity and linear combinations, and some solid geometry, especially lines and planes in 3 space.

Next, we asked the high school teachers at PCMI to discuss this with us. We hoped to learn what linear algebra topics were already present at the secondary level, how widespread was their inclusion in curricula, whether including the topics we identified was feasible, and, from the point of view of these teachers, whether it was desirable.

To our surprise, they reported that just about all of the topics we identified already appeared somewhere in the university preparatory curriculum, although they are often covered briefly or not at all, within an already crowded syllabus. They explained that the scope and emphasis of secondary mathematics is strongly shaped by the advanced placement (AP) exams in calculus. To oversimplify, the argument runs like this: parents and administrators are primarily concerned that able high school students succeed in passing AP calculus exams, so preparation for these becomes a driving force in shaping the curriculum. As long as solid geometry and linear algebra topics are absent from the AP exams, parents and administrators will have little sympathy for spending much time on these topics. The secondary teachers also pointed out that among all the students enrolled in advanced secondary mathematics courses, only a small fraction eventually take a college linear algebra course. Accordingly, it is difficult to justify including much linear algebra background material just to prepare a small number of students for a future college course.

After this session it was apparent that the topics we wished for will not become a standard part of the secondary curriculum any time soon. So what about students' prior knowledge is important for linear algebra teachers to consider? At the most concrete level, it is worthwhile to find out what linear algebra topics your students may already have seen. Have they worked with vectors, lines and planes in 3 space?

How many have already used matrix inversion on graphing calculators to solve linear systems? How many have worked with row operations? How do the answers impact your decisions about what to include in the course, and how long to spend on each topic?

Our discussions about the secondary curriculum were set in a larger context by a cross-program presentation given by William Schmidt, discussing his work [23] with TIMSS.[2] This led to discussion about the entire mathematics curriculum in the U.S. By now, the *mile wide – inch deep* metaphor will be readily familiar to the readers of the *Journal*. The notion that the syllabi of precalculus mathematics courses are already hopelessly fractured and busy was affirmed by the secondary teachers who participated in our discussions; for example they reported that no topic in high school mathematics can be studied in depth because only about a week can be devoted to each.

Thus our narrowly focused concerns on improving linear algebra instruction brought us into direct contact with issues of national scope in mathematics education: articulation between colleges and high schools; perceived needs for nationally adopted goals and standards for secondary education; criticisms of the entire framework of K-12 curricula; concerns about who should be taught, what they should be taught, and how the results should be assessed.

While these concerns do not bear directly on the teaching of linear algebra, they do deserve our attention. Every mathematics teacher is part of the much larger enterprise of mathematics education in this country. It is surely healthy for us to find out how our part of the system connects with other parts.

The Linear Algebra Student

Who are your students? What are their goals? Why are they taking linear algebra? Each UFP participant reported some kind of mixture of students from various populations: graduate school bound, prospective high school teachers, engineers and scientists, statisticians and social scientists. It was clear that our decisions about what topics to cover and emphasize have been predicated in part on what each of us believes our particular student population to be. What are your assumptions about the student mix? How do those assumptions influence how you teach the course, and are the assumptions justified?

Research on How Students Learn

Of all of the subjects we studied at PCMI, the issue of understanding how students learn was the one that made the greatest impact on the authors of this paper. From some initial doubts we initially brought to the program, we cultivated a full fledged skepticism that anyone, ourselves included, really understands

[2]Third International Mathematics and Science Study. See http://nces.ed.gov/TIMSS for information and publications.

what mathematics concepts students know or how they learned them. Let us qualify that. We believe you can tell pretty well when students have successfully learned what you wish them to learn about a specific topic at a specific time. But we doubt if anyone has a clear understanding of how students accomplish that, of the intermediate stages of understanding they may pass through along the way, of what their teachers did or did not do that assisted or impeded that learning, or of what long term understanding they will retain. These are significant questions, for many students do not master particular topics and harbor serious misunderstandings about some concepts.

There is not a great deal of published literature on how students learn linear algebra. Guershon Harel [9, 10, 11] has been studying some aspects of this for several years, and his papers provide some suggestions for linear algebra teachers. David Carlson [3] presents an interesting hypothesis about the special difficulties that linear algebra presents for students, and Ed Dubinsky [7] offers another point of view.

Looking further afield, there has been much more research on how young children learn basic mathematics topics. Although this research is not directly applicable to the linear algebra classroom, it provides clues that are, as Sherlock Holmes would have said, suggestive. We got some exposure to this kind of research at PCMI, both in informal discussions with the mathematics education researchers, as well as in lectures presented to the entire PCMI assembly.

One such presentation was made by Michael Battista [2]. He began with a filmed interview of a young girl, perhaps 7 years old. She was shown a diagram something like Fig. (1) and asked to predict how many square tiles would be needed to cover the rectangle. To answer this question, she used a strategy of first

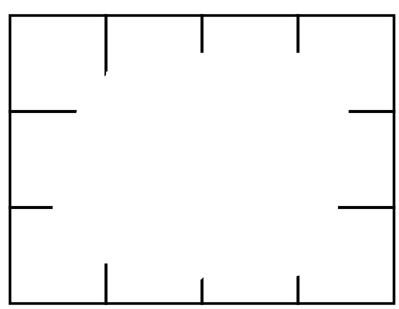

Figure 1: Tiling Exercise

counting around the perimeter of the diagram, and then estimating tiles in the center. She knew that there would be tiles in the center that had not been included in her tally, but clearly did not have a mental image of a rectangular array with a certain number of tiles in each row, and in each column. Later the child was given physical tiles and she successfully covered the diagram and counted out the 12 tiles that were needed. Finally, the tiles were put away, and the child was again confronted with the original task. She reverted to the perimeter strategy, and when she came to estimate the number of missing tiles from the center of the array, she was no more able to correctly envision the tiles than she had been at the start.

We came away from that presentation with two conclusions. First, we realized that something as concrete, as evident, as immediate, as the organization of a set of tiles into an array is in fact an abstract construction. To acquire it a child needs a certain mental maturity as well as practice and repeated exposure. Although it was completely obvious to us mathematicians, it was not obvious or natural for the child to conceptualize the arrangement of tiles as an array, even after she built the array herself out of physical tiles. Second, we obtained heightened convictions that there is more to learning mathematics than we sometimes imagine. It is naive to think that teaching consists of showing students clearly what you want them to know. This view completely ignores the cognitive development that can be required for even so simple an idea as the arrangement of tiles into rectangular arrays. The subject described above actually saw for herself how to arrange real tiles in an array. But that experience still did not make array arrangements a natural and obvious part of her enumeration tool box. It is hard to imagine that telling her about arrays would have produced significantly different results.

What does this mean, and what does it have to do with teaching linear algebra? By now, most teachers have heard the assertion that students have to construct their own knowledge in order to achieve meaningful learning. The presentation just described may help teachers understand better what it means to construct knowledge, and what such a construction might entail. To be more accurate, we are not sure how to assist students in this construction process, but it seems apparent that simply showing what is true and telling students what we wish them to know is not generally sufficient. Several careful studies in the context of calculus have demonstrated that students do not generally have a rich conceptual understanding of graphs and functions. We in the UFP agreed that linearity and independence are also concepts with which most students struggle. And it seems clear that simply doing a better job of telling and showing may not significantly improve their learning of such difficult topics.

What *can* we do? We asked the mathematics education researchers at PCMI for advice. The essence of their responses was that we should become better listeners. Specifically, they suggested that we select a few students and interview them in depth, instead of just quickly correcting their misconceptions. This means taking time to observe and listen carefully to what the individual student is really doing when he or she thinks about linear algebra. Explore what the student is thinking, paying careful attention to the ideas behind what the student says. How does a student reach conclusions that we find absurd? An organized way to proceed would be to ask for a few volunteers early who would be willing to be interviewed at regular intervals through your course. Keep notes. Over time, such interviews may reveal some general patterns in the thinking and misconceptions of our students that lead us to a better understanding of how they learn, although any such insights may well take several years to emerge. But in the short term, careful observation of an individual student has value as an end in itself, in helping that student perceive his or her misconceptions, and helping us to remember that the abstractions of linear algebra are *not* trivial and *do* require time and effort to acquire.

In our consideration of how students learn, we also discussed some very interesting work which has been going on in the physics community during the past decade. Assessment tools have been developed to measure the qualitative understanding of basic principles retained by students after they have studied college physics [13, 17, 24]. These assessment inventories have shown that even physics majors from prestigious schools still interpret everyday events in naive and incorrect ways; also, sometimes a student will answer essentially the same question differently, depending on how it is phrased. This information has already led to changes in the way physics is being taught.

Of course there are obvious differences between physics and mathematics. For one thing, physics is supposed to describe what goes on in the world of everyday experience in a way that mathematics does not. Moreover, students come to physics with well formed preconceptions of how the world operates, and these preconceptions are often at odds with Newtonian physics (e.g., they confuse force and momentum).

Fortunately for us, students are much less likely to show up in a linear algebra class with false perceptions about the subject. But the physicists' concern with meaningful learning is very similar to ours. A subgroup of the UFP participants made some very preliminary steps toward developing questions that might assess conceptual learning in linear algebra. The results of this effort are posted at the UFP website.

For the authors of this report, the experience of participating at PCMI inspired a rededication to helping students achieve meaningful learning, and a renewed respect for how difficult this can be. As mathematicians, we are aware of the rich interconnections of different ideas and concepts. We would like our students to learn how different ideas fit together, supporting and validating each other, and contributing to a deeper understanding of each part. We know from experience that understanding of this kind is not acquired as a result of being told of each individual fact and principle. It develops through actively exploring a mathematical topic, discovering and rediscovering the interconnections until they become utterly familiar and commonplace. But we who have developed understanding on this level risk forgetting the effort that came before: the missteps, false generalizations, incomplete and inconsistent conceptions. In full possession of the facts, we recite them piecemeal to our students as if we expect each new revelation to fit naturally and effortlessly into the pattern of what has already been presented. But teachers should be wary of this view, and skeptical of its effectiveness in the classroom. Meaningful learning is difficult to achieve and it rarely occurs unless students actively grapple with the ideas. We hope this article will inspire more discussion of how to encourage the right kind of grappling.

Teaching Methods

What can linear algebra teachers do to enrich, or replace, traditional lecturing, in order to better facilitate meaningful learning? Most of the UFP participants were familiar with various alternatives, including uses of technology, projects, and group activities. Some experiences with these ideas are recounted in [6]. The first author has observed Guershon Harel use an intense lecture-discussion method in his linear algebra classes at Purdue, which could be described as a rich extension of lecture. He insists that the students participate in working through all concepts. He uses MATLAB in a brief but important way, to facilitate examples: together the class figures out what must be calculated, how to do it and what results are expected; then he does the calculation and they discuss whether the results were what they expected and how to verify them.

Two of the UFP members rarely lecture at all; they have written Maple and Mathematica notebooks for all the topics in their linear algebra courses. They hold class in a computer laboratory and students work on these lessons individually or in groups, while the instructor moves around to assist as needed.

UFP participants discussed another variation on lecturing, related to the previously mentioned work by physics educators. An offshoot of that work is a method developed by Eric Mazur (see for example [21]; also visit the website http://galileo.harvard.edu, particularly under the heading of *Peer Instruction*). Mazur teaches basic physics at Harvard, in a large lecture hall, and uses the following quick but effective collaborative learning method. After he talks for about 15 minutes on a new topic, he displays a multiple choice question to find out if the students really understood that lecture. Their answers are quickly tallied and reported. Often there is substantial disagreement about what is the correct answer, so he asks them to discuss the question for 2-3 minutes and then vote again. The discussions are very lively, and typically result in correct responses from a large majority of the students on the second vote. When this happens, Mazur affirms that and goes on. Otherwise he discusses the students' misconceptions briefly. His students rarely miss class, for they have found out that these *put it together* questions are a key to passing the course!

Both authors of the present paper have already made use of Mazur's *polling* method in linear algebra classes. The first author, in particular, tried a few variations on the method. She used polling spontaneously and very quickly, to liven up lectures (*Show hands. Who thinks that idea will work? Who doesn't?*) She also used it more formally after many students missed the following question on an early test: *True or False? If A and B are invertible then $A + B$ is invertible.* Before returning the graded tests, she asked the class to vote on this question, and about half said *True*, half *False.* Then she asked students to find someone who disagreed with them and discuss which answer was really correct. She moved around, listening and occasionally asking pointed questions, for about 5 minutes. Then they voted again and about 80% got the correct answer. Some of the students who understood correctly explained how they went about answering it. This exercise took about 15 minutes, but it was a productive way to get students to think about how to analyze such a question, and to see how effective it can be to look for really simple examples.

Technology

Whether, when, and how to use technology is another question for linear algebra teachers to ponder. There are several different roles that technology can play in instruction, from eliminating computational drudgery in realistic applications, to providing environments for actively exploring the properties of mathematical structures and objects. The UFP participants had quite a variety of views and experience using MATLAB, Maple, Mathematica, Mathwright, and graphing calculators. Some of us assign computer projects to be done outside of class. As mentioned above, some use computer demos and examples to enrich lectures, and others rarely lecture at all, instead using software as a primary means for delivering mathematical material to the students, with a significant proportion of class time spent interacting with the computer. One participant, John Wicks, has written a textbook [25] intended for interactive use in conjunction with his Mathematica notebooks.

Once again we realized that different approaches have validity. We presented a workshop in which two representative linear algebra problems were solved using four different tools, Maple, Mathematica, Matlab and a graphing calculator. Information on this activity, with detailed instructions for using the different tools, are provided at the UFP web site.

Here we will restrict ourselves to summarizing our discussions about how and why to use software in teaching linear algebra. The main purposes we identified are: for computation in meaningful applications; as a matrix calculator; as a direct focus of instruction; for visualization; to provide an environment for active exploration of mathematical structures; and to explore some of the limitations of floating point calculations.

It is not difficult to find realistic applications that will be of interest to students from just about any background. However, in most real world problems, the dimensions of the matrices make hand calculation completely inadequate. Even with relatively low dimensional problems, the overhead of hand calculation quickly becomes distracting or simply overwhelming. Some teachers use technology just to provide students first-hand experience with real applications in realistic settings.

Calculators and software like Matlab, Maple, and Mathematica provide students a means of instantly and effortlessly performing matrix computations, and thus free them to concentrate on what the computations mean, and when and why to perform them. Many instructors use software in this context. The focus is not necessarily on realistic applications. Rather, students are intended to answer questions about what happens when certain computations are performed, without having to think too much about the mechanics of carrying out the operations. For example, students might experiment with the effect of scalar or diagonal matrices as multipliers, without actually performing all the matrix multiplications by hand. Most instructors feel that doing *some* of the matrix multiplications by hand provides insight about why results appear as they do. But many also believe that the ability to rapidly investigate a large number of examples makes a contribution to understanding.

Software can provide helpful visualization with two and three dimensional graphics. The ATLAST project provides a number of excellent tools. For example, the program *span* plots in 3 space the images of a large number of vectors (essentially chosen at random), under multiplication by a particular fixed matrix. By rotating the display, students can quickly see whether the plotted points appear to lie on a line or plane. Some other suggestions are presented by Hern and Long [12].

Software can be used to create interactive environments in which students can explore and experiment with vectors, matrices, transformations, etc., with graphical, symbolic, and numerical representations. Usually students must learn some syntax to use the software, but it is possible to free them from that by creating activities with a windows-style point and click interface. Matlab supports this kind of development, and to a more limited extent, Maple and Mathematica can be used in a similar way. Web pages permitting interactive exploration can also be created using JAVA, although that topic was not explored at PCMI. For creating activities of this kind, the most powerful tool of which we are aware is Mathwright (see reviews [8, 18]). There is a version of Mathwright that is available for free on the Internet, along with sample activities for students – see http://www.mathwright.com/. The library includes a few linear algebra activities.

Finally, because floating point computation plays a huge role in the applications of linear algebra, some teachers believe that some brief discussion of topics like operation counts, matrix factorization results, and numerical stability and accuracy of the computational algorithms are important in a first linear algebra course. This may be the only place where undergraduates encounter the potential pitfalls of machine computation. As ski instructors know well, the first lesson is about how to stop!

Conclusions

Who knows how to teach linear algebra? We all do, and none of us do. We all do in the sense that we all have a pretty good idea what we will do the next time we are scheduled to teach that course. But we all need to understand better how students learn, and to recognize that the appropriate content, methods and context will be different in different settings. There is no one *right* way to teach that course, and there are issues that may never be definitively resolved. By participating in the UFP, we found new questions to ponder, and reasons to look at old familiar questions in new ways. We hope that this article has had a similar effect on the reader, and that the references and UFP web site may provide resources for further study.

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