Calculus Notes For Review Sessions I and II to be held on 02/05/2011

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Introduction

These notes have been created to complement material presented in the short review session focused on understanding calculus. This review session will be divided up into four two hour meetings for a total of eight hours. Due to the limited number of class hours, it would be impractical to expect to walk away with a complete understanding of this important branch of mathematics. Instead, you should expect to walk away with a better understanding of concepts and an insight to how different concepts are related. My personal feeling is that a lot of math classes brush over these insights and instead focus on solving problems. This is especially true in calculus classes designed for non-math majors in undergraduate education. There is a good reason for this though. Understanding the concepts requires effort, but the payoff is a much deeper understanding of common concepts and a better foundation to learn more advanced concepts. Therefore, this class will not focus on solving problems, we will focus on understanding the basic concepts that any solution in calculus relies upon. We will solve some problems, but when we do, the focus will be on understanding why the solution makes sense relying on our understanding of the basics.

Mathematics is an exercise in logical reasoning. Most people are not aware of the importance of logic in mathematics. Math starts with a few very basic assumptions (called axioms). These are the important and necessary building blocks of each math concept we will see in this class. Axioms are important because they are just accepted to exist and be true. All of the concepts we will see in this class are based on a very logical argument for why they are true. Behind any result or concept in math lies a proof. The proof is a sequence of deductions starting from some basic assumptions. The beauty of a proof results in the logic and its connection to other proven concepts in math. This class will not use proofs, but it is important to at least have some understanding that there is a lot of work behind the concepts that we will freely use. There are some advanced math classes that start from the very basics and spend a whole semester without seeing a derivative or integral since it is impossible to rely on something before you prove its existence. This is true for many of the concepts like sequences, limits, derivatives, and integrals that will be used in this class.

1This is from personal experience. I started my undergraduate education in a ‘calculus for biology’ class. In fact the teaching assistant on the first day told the whole class, ‘since you are not taking the ‘real’ calculus class, then you must have no interest in ‘real’ math, therefore I am not going to waste my time explaining the insights, we will just do problems so you can pass the midterm and final.’ That was the single worst education experience I have had to endure. The goal of this class is to avoid this attitude.
Calculus is a very important branch of mathematics since it focuses on change. Calculus involves a lot of new concepts and expressions which make it an intimidating subject to study. From personal experience, I started my understanding of more advanced math near the end of my undergraduate education since I felt intimidated by the subject. There are good reasons why we use such expressions and the point of this class is to understand each expression. You are encouraged to ask questions about anything in these notes or material presented in class. It is not like an English class, where it is not really necessary to know the definition of each word. Often you can get by from the context surrounding the word. This is often not true for math and especially calculus. It is necessary to know how to define the concepts or be able to draw a picture. If you find yourself puzzled by a problem or question, then there is a good chance that you are missing a basic point.

As a final note before we start this class, a math professor once told me that if you can not solve a problem, then that just means that you are not looking at the right problem. Instead, you should break down the problem into a series of smaller problems. You break down the problem until you have an answer and then take this answer and address the larger problems. Eventually, you work your way back to the original problem and you will have a solution. This is very wise advice, especially when it comes to math. Many people see a problem and stop in frustration and never make any real progress. If you find yourself in this situation, put away the problem for awhile, and then when you return, break down the problem. You will likely find that the problem is some unclear definition. This advice also helps remove the thought that there is something wrong with you and instead you realize that there is something wrong with the problem!

These notes will review the primary concepts that will be used in this class. We will start with the more basic concepts and emphasis is on clear definitions. For a good personal test, after you review a section, you should try to cover these notes with a blank piece of paper and write out the definition for the main concept shown in bold type. Keep in mind that sometimes, it might be more useful to draw a picture instead of using words.

These notes are provided to review in detail some of the concepts that will be introduced in class. These notes are not meant to replace the lectures. You should review these notes before the class. If there are parts of these notes that you find confusing, do not worry, we will discuss them in more detail in class. You will get more from the lectures if you read these notes. If you have questions then make sure and bring these up during the lecture. You can also email these questions to jroberts@arec.umd.edu.
Functions

A single variable function, \( y = f(x) \), is just a rule that states the value of \( y \) is going to depend on the value of \( x \). We use letters in math to identify the variables, or parts of the function that are allowed to vary over some set of numbers. The ‘functional form’ is the actual rule and will also involve constants or parameters which are fixed if we are referring to a particular function. The constants or parameters are often also shown as letters if we wish to refer to a general functional form. It is important to always keep in mind which values of the function will be fixed and which values will be allowed to vary. It is a very important difference which we will discuss in more detail below. When you see this kind of expression, you should refer to this as ‘\( y \) is a function of \( x \)’. Even though the complete expression has two variables, there is only one variable on either side of the equal statement. Given this expression, there are specific ways to refer to the different variables:

- \( y \) is the outcome or dependent variable (the outcome \( y \) depends on \( x \))
- \( x \) is the explanatory or independent variable (\( x \) explains \( y \))

It is important to realize that we define \( f() \) and then we specify the values of \( x \), which will determine the set of \( y \) values. The values of \( x \) are often going to be either the full number line (\( -\infty \) to \( +\infty \)) or some portion such as \( (0 \) to \( +\infty \)) or \( (0 \) to \( 5 \)). The values of \( x \) are going to depend on the problem. For example, if we specify a function that explains how height is related to age, height = \( f(\text{age}) \), then it wouldn’t make sense to have our \( x \) values be negative. Regardless of the endpoints, calculus is built on the notion that there are an infinite number of values in this range. This is true even if the range is between 0 and 1. We will see why this is important soon.

It is important to start to recognize how function must look given how we define \( f(x) \). For the functions listed below, take a piece of paper and draw a cross in the middle of the page. Label the horizontal line \( x \) and label the vertical line \( y \). Pick a few numbers for \( x \) and then solve for \( y \), then plot the \((x, f(x))\) point on the figure.

- \( y = f(x) = x + 1 \)
- \( y = f(x) = 2x \)
- \( y = f(x) = 2x + 1 \)
- \( y = f(x) = -2x - 1 \)
Each of these functions are linear functions which has the general functional form of $y = f(x) = mx + b$. This is a general expression for a line. It is important to note that $x$ is still the only independent variable in this function. The function has parameters $m$ and $b$. Each of the examples above are specific linear functions and are selected from the full set of all possible linear functions. We refer to this full set as the ‘family’ of functions defined by the general functional form. A linear function is an important example of a special kind of function. Given this functional form, if you were to tell me the value of $y$ or $f(x)$, I would be able to tell you precisely the value of $x$ that generated this $y$ value. As we will see below, this is not always possible. This will be discussed in more detail in class.

An inverse function is simply the rule rearranged to specify $x = g(y) \equiv f^{-1}(y)$, which means that we have reversed the roles of the variables. Now $x$ is a function of $y$. This is an easy task given the special property just discussed. For example, take a look at the following inverse functions and note how they relate to the original functions.

- $x = f^{-1}(y) = y - 1$
- $x = f^{-1}(y) = y/2$
- $x = f^{-1}(y) = y/2 - 1$
- $x = f^{-1}(y) = -y/2 + 1$

**Question 1:** A function is increasing if $x_1 > x_2$ implies that $f(x_1) > f(x_2)$. Find the values on the parameters above such that we have an increasing function. Why might it be important to know if we are dealing with an increasing function? Can you find the condition that implies we have a decreasing function? Think or draw some possible functions that are either always increasing or always decreasing.

**Question 2:** The parameter $m$ for a linear function has a special interpretation. Can you explain the significance of $m$ as defined in the expression above? Explain the importance of $m$ as we increase or decrease the value of $x$ by one unit.

Below are a few examples of other functions. Note how these functions compare to the linear functional form. For each of these functions it is important to be able to determine how they should behave. The first function listed below has the variable $x$ raised to the second power. The value of this exponent is important given we allow $x$ to be equal to any possible value. Note that $f(x) = f(-x)$. This is not true for the second function listed below.
• $y = f(x) = x^2$
• $y = f(x) = x^3$
• $y = f(x) = 2x^2$

**Question 3:** Each of the functions listed above have a general functional form, $f(x) = ax^b$, where $a$ and $b$ are parameters. If $x = 1$, then $y = a$. Think about what the values of $a$ and $b$ imply for how this value of $y$ changes as we increase $x$ by 1. Then consider increasing $x$ by 10 and 100. Contrast this against the findings in **Question 2**.

For **nonlinear functions**, the slope is a function of the variable $x$. This will be explained in detail during class.

**A two variable function**, $z = f(x, y)$ is also a rule, but now the value of $z$ is going to depend on the values of $x$ and $y$. It is important to realize that if we fix one of the variables (e.g., $y = 5$), then we are left with, $z = f(x, 5)$, which is a single variable function. We can then draw this as above noting that it is conditional on $y$ being held fixed. This is going to be very important when we look at multivariable calculus. There are a lot of different possibilities for the rule, below we write some general functional forms. For each example, the variables are $z, x,$ and $y$. Any other letter is a parameter.

• $z = f(x, y) = ax + by + c$
• $z = f(x, y) = ax^b + cy^d + e$
• $z = f(x, y) = ax^bcy^d + e$

The first function shown above has an important connection to the linear function presented above. Take a while and compare these two functional forms. You will find that if we fix $y = 3$, then we are left with $ax + b3 + c$ which since we can always redefine parameters, can be made to look like $m_1 x + b_1$ where now $m_1 = a$ and $b_1 = b3 + c$. This is a simple way to demonstrate what was discussed above. The first function is the line extended to a higher dimension, which is a plane, a flat surface in 3d. To visualize this, take a piece of paper and allow this to hover over your desk. Your desk is the $(x, y)$ space and the plane lives in the third dimension.

In math, as you add variables, you are adding dimensions. A single variable is only a dot and can be shown on any line. If we have two variables, then the function defines how these two variables are related in 2d
space. If we have three variables, then we have something in 3d space or a surface. If we have four variables or higher, then we are in some sort of space that makes it very difficult to visualize the function, but as we will see there are tools to help understand how the function behaves. This is why it is important to be very sure which letters in a function are variables and which letters are parameters.

Likewise, we can define a multivariable function, \( z = f(x_1, x_2, x_3, \ldots x_n) \). In this expression there are \( n \) variables. Just as above, if we fix the first \( n - 1 \) variables, then we will once again be left with a single variable function.

During the class we will review some different functions and I will use some computer software to help visualize the function.

There are some special functions that you will see often. These include \( \exp(x) \) and \( \ln(x) \). The exponential function is the function \( e^x \) or \( \exp(x) \) where \( e \) is the number approximately equal to 2.71828. There are specific reasons why one might want to use this functional form which we will discuss in more detail in class. The natural logarithm function is the function \( \ln(x) \) or the logarithm of \( x \) to the base \( e \). The connection between the two is the fact that the inverse function of \( e^x \) is the natural logarithm function \( \ln(x) \). These functions will be used a number of times in the class.

We will spend the all of session I on functions.

Limits

Before discussing derivatives, we need to review the concept of a limit. If you took a calculus class then you probably have bad memories of learning about limits using \( \delta \) and \( \epsilon \) characters. Limits are often the first real abstract notion that is presented in calculus and often is the most intimidating aspect of learning calculus. They are essential to calculus though and we use limits to define important concepts such as continuity, derivatives and integrals. We need to have an understanding of what a limit is before we proceed.

Recall above that when we define a function, \( y = f(x) \), we are going to have some set of \( x \) values that we are going to feed into the function to produce the set of \( y \) values. In this context, the possible values of \( x \) is referred to as the domain of the function. The resulting values of \( y \) is referred to as the range of the function. If we have a general linear function, then if the domain is \((-\infty \text{ to } +\infty)\) then the range will also
be $(-\infty, +\infty)$. Now, recall also that we stated above that any interval in the domain is going to have an infinite number of points. This is necessary here since when we talk about limits we are going to be taking about very small intervals and more importantly, these intervals are going to get smaller and smaller, so it is important that there is always going to be numbers in these intervals regardless of the size. Take a moment to visualize this since this will reinforce the concept of a limit. Fix a point, for example, let $x = 1$.

Now think of the interval $(.5, 1.5)$ which surrounds $x = 1$, then think of the interval $(.75, 1.25)$ which again surrounds the initial value and furthermore is completely contained in the first interval. We can now just redefine this by letting $\delta_1 = .5$ and $\delta_2 = .25$ with the corresponding intervals $(x - \delta_1, x + \delta_1)$ and $(x - \delta_2, x + \delta_2)$.

Now, the limit is a property of the function $f(x)$ which is equal to $y$, our outcome or dependent variable. The limit of a function is a value which the function approaches as the input or independent variable approaches some value. As discussed above, as we continue to take smaller and smaller intervals around $x = 1$ we are essentially letting $\delta$ go to zero. This is done to determine how the function behaves and determine what value of the function we approach during this process. It is important to realize that the limit is an approximation. In practice, we can never truly observe a limit, but depending on the functional form, often this is not required. All we need is to be confident that what we are approaching is going to remain. We will present some examples in class. You should note that the limit is a very important concept in calculus and we could spend 8 hours along just talking about limits and their importance. This is only a very short overview.

In class we will discuss in more detail a few examples and show why the notion of a limit is necessary for the work that will follow. We will spend the first half of session II on limits.

**Derivatives**

We will see derivatives a lot in this class. These are NOT the derivatives you have likely heard a lot about on the news recently. The derivative in calculus has a very specific meaning. It is very important to be very clear how you use this term and make sure you are using it in the correct context. Recall from above a function, $y = f(x)$, has an input $x$ and an output $y$ equal to the value of the function at a given level of $x$. Recall also in the introduction that we stated calculus is the study of change. The derivative is a measure of how a function changes as its input changes. It is important to think of a linear function. Note this function has the same slope at any value of $x$, so we should expect that if we increase $x$ by one unit, then the change
in the function should be a constant regardless of the initial value of \( x \). Now, think of a linear function, but add a curve to it. Now the value of the derivative will depend on the value of \( x \). You should spend some time to make sure this is clear and try to draw this on a piece of paper.

The derivative by itself has a very specific meaning. Often though you are trying to tell a story with a specific functional form. For example, if the input represents age and the output represents height, then the derivative takes on a function specific meaning. You are now concerned with how height changes given a change in age. Note that this is a nonlinear function since at some age we stop growing.

**Question 4:** In the example presented above, imagine you start at age ten and at age 40. If at each point you increase the age by one, how do you expect the following derivatives to compare?

We will spend the second half of session II and all of session III on derivatives.