A note on FRA's.

Introduction.

A Forward Rate Agreements (FRA) is nothing more than a contract whereby two parties lock in a forward interest rate. To understand FRA's, it is therefore necessary to first review the forward interest rates that are implied by the term structure of interest rates.

A. Implied Forward Rates.

Definition #1.

Consider two deposits made at time $t$. They are similar in all respects except for their maturities and interest rates. The first deposit matures at time $t+n$ and bears interest rate $r_{bid}(t,n)$. The second deposit matures at time $t+T$ and bears interest rate $r_{bid}(t,T)$, with $n<T$. The implied forward bid rate from time $t+n$ to time $t+T$ is simply the interest rate $IFR_{bid}(t+n, t+T)$ such that, if the earlier-maturity deposit were reinvested at the rate $IFR_{bid}(t+n, t+T)$ from time $t+n$ to time $t+T$, the final yield on this roll-over of short-term deposits would equal the yield on the longer-maturity deposit.

Now consider instead two loans made at time $t$. They are similar in all respects except for their maturities and costs. The first loan matures at time $t+n$ and requires interest rate $r_{ask}(t,n)$. The second loan matures at time $t+T$ and requires interest rate $r_{ask}(t,T)$, with $n<T$. The implied forward asked rate is simply the interest rate $IFR_{ask}(t+n, t+T)$ such that, if the earlier-maturity loan were rolled-over at the rate $IFR_{ask}(t+n, t+T)$ from time $t+n$ to time $t+T$, the final yield on this roll-over of short-term loans would equal the yield on the longer-maturity loan.

Definition #2.

When defining the IFR in the paragraph above, we concentrated either on the deposit term structure or on the loan term structure, but not on both. An alternative view of IFR's is the following. The $IFR_{bid}$ is the rate such that, if we borrow long and lend short, we will break even if the short loan is rolled-over (i.e., if we lend again) at the implied forward rate. Similarly, the $IFR_{ask}$ is the rate such that, if we borrow short and lend long, we break even if we refund the loan by borrowing again at the implied forward rate.
Notice that the rates thus defined are slightly different from those obtained in definition 1. As long as the bid-ask spreads on the (cash) money market are small, however, the effective difference between the two definitions will be small (see, in particular, the numerical example in Section B.4 below).

B. Forward Rate Agreements.

1. Definition.

Under an FRA, two parties essentially agree to lock in an implied forward rate. It works in the following way. An FRA is a (cash-settled) contract by which the seller agrees to pay the buyer, at a given point in the future, the increased interest cost on a nominal sum of money if the interest rate at that time has risen above an agreed-upon interest rate, but the buyer agrees to pay the seller the increased interest cost if the interest rate at that time has fallen below the agreed-upon rate. That is, the long party gains from the FRA if interest rates go up (precisely, if the spot rate at maturity is greater than the FRA rate) and the short party wins if interest rates drop.

For instance, a "6 against 9" FRA would lock in the 3-month interest rate that applies to deposits starting at the end of the 6th month after the contract was signed and ending at the end of the 9th month after it was signed. The FRA cash settlement would take place at the end of the 6th month:

\[
\text{today} \quad 6 \text{ months} \quad 9 \text{ months} \\
\text{(FRA signed)} \quad \text{(cash settlement)}
\]

The exact amount of the cash-flow taking place between the two parties is given by:

\[
\text{amount paid by the FRA seller} = (\text{nominal amount of contract}) \times \frac{(S-A) \times \left(\frac{\text{# days the FRA runs}}{\text{# days in the year}}\right)}{1 + S \times \left(\frac{\text{# days the FRA runs}}{\text{# days in the year}}\right)}
\]

where, A is the rate agreed on in the FRA and S is the market interest rate that prevails at the time the FRA cash settlement takes place.
2. Effective borrowing costs or deposit rates.

J.P. Morgan sells a "3 against 12" FRA for $1m at an annualized rate of 4.75%. Three months after the sale, interest rates have the following term structure:

<table>
<thead>
<tr>
<th>maturity (# months)</th>
<th>annualized rate(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>4.5</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Q1: How much cash does the bank pay to, or receive from, the FRA buyer?

A: $1,807.23. By selling the FRA at 4.75%, JP Morgan wanted to make sure that it would obtain a 4.75% annualized rate on a $1m 9-month loan it would make 3 months later. Since, 3 months after the FRA sale, the 9-month rate has become 5%, JP Morgan in fact can lend at 5%.

Since 5% > 4.75%, JP Morgan will pay the interest rate differential to the FRA buyer on the nominal amount of the contract. The exact cash settlement, 3 months after the FRA sale, is:

\[
\text{amount paid by the FRA seller} = \text{(nominal amount of contract)} \times \frac{(S-A) \times \left(\frac{\# \text{ days the FRA runs}}{\# \text{ days in the year}}\right)}{1 + S \times \left(\frac{\# \text{ days the FRA runs}}{\# \text{ days in the year}}\right)}
\]

\[
= (\$ 1m) \times \frac{(.05 - .0475) \times \left(\frac{270}{360}\right)}{1 + .05 \times \left(\frac{270}{360}\right)}
\]

\[
= \$ 1,807.23
\]

Q2: What is J.P. Morgan's effective lending rate for the 270-day lending period?

A: 4.75%.

By entering into the FRA agreement, JP Morgan has ensured that, regardless of the actual 9-month rate that will prevail 3 months after the FRA sale, it would receive 4.75% on money that it would lend for 270 days: if the cash rate 3 months after the FRA sale were higher than 4.75%, then JP Morgan would pay the interest difference to the FRA buyer; and if the cash rate were lower, then it would receive the interest difference from the FRA buyer.
3. Arbitrage strategies using FRA's.

Suppose that the following interest rates are observable today on the market. "Cash" stands for loans/deposits that require cash movements today.

<table>
<thead>
<tr>
<th>Cash</th>
<th>FRA &quot;6 against 9&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bid</td>
</tr>
<tr>
<td>6-month</td>
<td>10 8/16</td>
</tr>
<tr>
<td>9-month</td>
<td>10 10/16</td>
</tr>
</tbody>
</table>

Given those rates, there are two arbitrage strategies to consider:

(i) borrow short at 10 9/16%, roll over at FRA ask of 10.58% and invest long at 10 10/16% (i.e., at 10.625%). Given that 10 10/16 is larger than both 10 9/16 and 10.58, your return on investment is clearly higher than the borrowing cost each period and there is an arb opportunity.

(ii) borrow long at 10 11/16%, invest short at 10 8/16% (i.e., at10.5%) and roll over the short-term deposit at the “6 against 9” FRA bid of 10.48%. Investing at the 6-month cash rate and rolling over at the bid FRA rate, would yield a rate of return of less than 10.5% (since the rate you’d get is the weighted average of 10.5% and 10.48%). Clearly, there is no arb since 10 11/16% > 10.5%.


Suppose that the following interest rates are observable today on the Eurodollar market:

<table>
<thead>
<tr>
<th>Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>6-month</td>
</tr>
<tr>
<td>9-month</td>
</tr>
</tbody>
</table>

What is the range of bid or asked prices (i.e., rates) that a bank could quote its customers for 6 against 9 FRA's. (Assume that all months have 30 days, and that the year has 360 days).

**Answer.**

The bank, as a market maker, faces two sets of constrains: not offering opportunities for others to arbitrage against the bank, and being attractive. Formally:
(i) no-arbitrage conditions.

First, if the FRA bid rate is too high, then arbitrage opportunities will arise. Put differently, it must not be profitable for investors to borrow at the 9-month rate in order to then lend at the 6-month deposit rate and roll the deposit over at the FRA bid. Hence, we must have:

\[
\left(1 + \frac{9^{8/16}}{2}\%\right) \left(1 + \frac{\text{FRA}_{\text{bid}}}{4}\right) \leq \left(1 + \frac{3 \times 9^{11/16}}{4}\%\right)
\]

that is, \( \text{FRA}_{\text{bid}} < 9.606\% \).

Similarly, if the FRA asked rate is too low, then arbitrage opportunities will arise. Put differently, it must not be profitable for investors to lend at the 9-month rate by first borrowing at the 6-month cash asked rate and then rolling the loan over at the FRA asked rate. Thus, we have:

\[
\left(1 + \frac{9^{9/16}}{2}\%\right) \left(1 + \frac{\text{FRA}_{\text{ask}}}{4}\right) \geq \left(1 + \frac{3 \times 9^{10/16}}{4}\%\right)
\]

that is, \( \text{FRA}_{\text{ask}} > 9.305\% \).

(ii) competitive pressures.

To calculate the bid FRA rate, notice that no one will be interested in depositing money for 6 months and rolling the deposit over at the bid FRA unless, by doing so, they obtain at least as good a rate as they would by depositing the money at the 9-month rate. Thus, we have:

\[
\left(1 + \frac{9^{8/16}}{2}\%\right) \left(1 + \frac{\text{FRA}_{\text{bid}}}{4}\right) \geq \left(1 + \frac{3 \times 9^{10/16}}{4}\%\right)
\]

which requires that \( \text{FRA}_{\text{bid}} > 9.427\% \).
Combining this restriction with the no-arbitrage restriction yields: $9.427\% < FRA_{bid} < 9.606\%$.

To calculate the asked FRA rate, notice that no one will be interested in borrowing money for 6 months and rolling the loan over at the asked FRA unless, by doing so, they obtain at least as good a rate as they would by borrowing the money at the 9-month rate. Thus, we have:

$$
\left(1 + \frac{9^{9/16}}{2}\% \right) \left(1 + \frac{FRA_{ask}}{4}\% \right) \leq \left(1 + \frac{3 \times 9^{11/16}}{4}\% \right)
$$

which requires that $FRA_{ask} < 9.484\%$.

Combining this restriction with the no-arbitrage restriction yields: $9.305\% < FRA_{ask} < 9.484\%$.

(iii) quotes.

We can now conclude. Remembering that asked rates must be higher than bid rates so as to avoid offering a money machine to customers, we obtain:

$$9.427\% < FRA_{bid} < 9.606\% \quad \text{and} \quad 9.305\% < FRA_{ask} < 9.484\%$$

which yields:

$$9.427\% < FRA_{bid} < FRA_{ask} < 9.484\%$$

as the range of possible quotes.


a. Suppose that a bank in London quotes the following rates:

<table>
<thead>
<tr>
<th></th>
<th>FRA &quot;6 against 9&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bid(%)</td>
<td>Asked(%)</td>
</tr>
<tr>
<td>9.48</td>
<td>9.58</td>
</tr>
</tbody>
</table>

Are there arbitrage opportunities at those rates? Explain.

Answer.

There are no arbitrage opportunities at those rates:

+ since the ask quote $= 9.58\% > 9.305\%$, one cannot borrow at the 6-month cash asked rate and roll over at the asked FRA, invest the proceeds at the 9-month bid rate, and still profit;
+ since the bid quote = 9.48% < 9.606% (indeed, <9.484%), one cannot borrow at the 9-month cash asked rate, lend at the 6-month cash bid rate and roll over the loan at the bid FRA, and still profit.

b. Suppose that you are Assistant to the Treasurer at IBM. Your boss tells you that "Big Blue" anticipates to enjoy excess liquidities in the amount of $50m between 6 and 9 months from now, and plans to invest them in the Eurodollar market (in the form of dollar-denominated euro-CD’s). Based on the rates above, would you recommend to her to lock in the rate using an FRA or to wait 6 months and take her chances on the then-prevailing spot rates?

Answer.

First, recall the term structure of deposit rates (cash):

<table>
<thead>
<tr>
<th>Bid</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-month</td>
</tr>
<tr>
<td>9-month</td>
</tr>
</tbody>
</table>

The 6-against-9 forward rate implied by the term structure above is given by:

$$\left(1 + \frac{.09^{8/16}}{2}\right) \left(1 + \frac{\text{FRA}_{\text{bid}}}{4}\right) = \left(1 + \frac{3 \times .09^{10/16}}{4}\right)$$

that is, $\text{FRA}_{\text{bid}} = 9.427\%$.

Since the quoted (i.e., actual) FRA bid rate is higher (9.48%) than the implied (i.e., theoretical) forward bid rate (9.427%), you should make the following recommendation:

- if the “long” rate (the 9-month cash rate of $9^{10/16}$ in this case) does not reflect any liquidity premium, then (assuming markets are efficient) the IFR of 9.427% is just what the market expects the 3-month cash (i.e., spot) rate to be in 6 months. Since the FRA rate bid by the dealer is quite a bit higher (9.48% vs. 9.427%), IBM should sell an FRA to lock in the deposit rate it will get in 6 months.

- If money-market investors have a preference for liquidity (i.e., if we believe that investors generally need to be enticed to hold longer-dated instruments (in this case, 9-month CDs) rather than shorter-dated ones (in this case, 6-month CDs), then the expected 3-month rate in 6 months is even lower than 9.427% (precisely, it is equal to 9.427% minus the liquidity premium, whatever the latter is). In this case, you should recommend even more forcefully that IBM go short (“sell”) a $50m FRA.