Derivatives

• Lectures 10-12:
  • Part VI: Option Pricing
    » Black-Scholes Formula
    » Binomial Trees

• Last two topics (Weeks 13-14):
  • Part VII: Option pricing & Continuous-time Finance
    » Modeling Asset Prices in Continuous Time
    » Derivation of the Black-Scholes Formula
  • Part VIII: Option pricing – Advanced Topics
    » Hedging; Option Pricing w/ varying risk-free rate

Part VII:
Option Pricing
&
Continuous-Time Finance
Modeling Stock Prices Behavior

- Stochastic Process & Price Modeling
  - continuous time
  - continuous variables
- Markov Processes
  - market efficiency
- Wiener Processes
- Ito Processes & Ito’s Lemma

Stochastic Process & Price Modeling

- Dimensions
  - time
    - discrete measurement intervals vs. continuous flow
  - random variable
    - limited number vs. interval of possible values
- Types of Stochastic Processes
  - Discrete time; discrete variable
  - Discrete time; continuous variable
  - Continuous time; discrete variable
  - Continuous time; continuous variable
Stochastic Process & Price Modeling 2

- Our focus
  - We model stock prices
    *with* continuous-time, continuous-variable process
  - Why?
    » most useful type of process
    *to value* derivative securities

From Discrete to Continuous Time

- Normal distribution
  - type of process
    » Discrete time, continuous variable
  - example
    » stock price is currently at $40
    » at the end of 1 year, probability distribution
    » normal distribution \( \phi(40,10) \), with
    \( \phi(\mu,\sigma) \):
      \( \mu \): mean
      \( \sigma \): standard deviation
From Discrete to Continuous Time 2

• Taking limits
  » probability distribution of the stock price at the end of
    - 2 years?
    - ½ years?
    - ¼ years?
    - Δ years?

• As Δ → 0,
  » we have defined
  » a continuous variable continuous time process

Markov Processes

• Basic idea
  • a stochastic process “is Markov”
  • if future movements in the random variable
    » depend only on where we are
    » do not depend on the history of how we got there

• Relevance
  • all 4 types of stochastic processes can be Markov

• Our approach
  • assume that the underlying asset’s price
  • follows a Markov process
Markov Processes 2

• Weak-Form Market Efficiency
  • definition
    » impossible to produce consistently superior returns
    » with a trading rule
    » based on the past history of stock prices
  • interpretation
    » technical analysis does NOT work
• Observation
  • a Markov process for stock prices
  • is consistent with weak-form market efficiency

Markov Processes 3

• Variances & Standard Deviations
  • changes in successive periods of time
    » independent in Markov processes
  • implication
    » variances are additive
    » standard deviations are NOT additive (why?)
  • example
    » “in 1 year, probability distribution is \( \phi(40,10) \)”
      - correct to say that variance is 100% per year
      - strictly speaking, NOT correct to say that standard deviation is 10% per year
Wiener Processes

• Definition
  • a variable \( z \)
    » whose value changes continuously
    » s.t. the change in a small interval of time \( \Delta t \) is \( \Delta z \)
  • follows a Wiener process if
    » 1. \( \Delta z = \varepsilon \sqrt{\Delta t} \)
      where \( \varepsilon \) is a random drawing from \( \phi(0,1) \)
    » 2. the values of \( \Delta z \)
      - for any 2 non-overlapping periods of time
      - are independent

Wiener Processes 2

• Properties of a Wiener Process \( z(t) \)
  – moments
    • Mean of \( [z(T) - z(0)] \)
      » = 0
    • Variance of \( [z(T) - z(0)] \)
      » = \( T \)
    • Standard deviation of \( [z(T) - z(0)] \)
      » = \( \sqrt{T} \)
Wiener Processes 3

• Taking Limits
  • what?
    \[ \Delta z = \varepsilon \sqrt{\Delta t} \] becomes \[ dz = \varepsilon \sqrt{dt} \]
  • interpretation
    » the corresponding expression involving \( \Delta z \) & \( \Delta t \)
    » is true in the limit as \( \Delta t \) tends to zero
    » This is how we define the meaning of \( dz \) & \( dt \)

• stochastic calculus
  » in this respect,
  » stochastic calculus is analogous to ordinary calculus

Generalized Wiener Processes

• Wiener process
  • drift rate ("average change per unit time") = 0
  • variance rate = 1

• Generalized Wiener process
  • idea
    » drift rate & variance rate = any chosen constants
  • definition
    » the variable \( x \) follows a generalized Wiener process
      with drift rate \( a \) & variance rate \( b^2 \)
    » if \( dx = a \, dt + b \, dz \) where \( dz \) is Wiener
Generalized Wiener Processes 2

\[ \Delta x = a \Delta t + b \epsilon \sqrt{\Delta t} \]

- Mean change in \( x \) in time \( T \)
  \[ = a T \]
- Variance of change in \( x \) in time \( T \)
  \[ = b^2 T \]
- Std deviation of change in \( x \) in time \( T \)
  \[ = b \sqrt{T} \]

Generalized Wiener Processes 3

- Our Example, Revisited
  - data
    - stock price \( S \) starts at $40
    - proba. distribution is \( \phi(40,10) \) at the end of the year
  - if stochastic process for \( S \) is Markov with NO drift
    - then the process is \( dS = 10dz \)
  - if the stock price is expected to grow by $8 on average during the year
    - then the year-end distribution is \( \phi(48,10) \)
    - and the process is \( dS = 8dt + 10dz \)
Generalized Wiener Processes 4

• Problem
  • what?
    » generalized Wiener process NOT Appropriate for stocks
  • why?
    » for a stock price, what is constant in a short period
      - is likely to be the expected proportional change
      - is NOT the expected absolute change
    » for a stock price, uncertainty
      - as to the size of future stock price movements
      - is likely proportional to the stock price level

• Solution
  • Ito processes

Ito Processes

• Characteristic
  • drift rate & variance rate are functions of
    » time
    » & the variable
    \[ dx = a(x,t)dt + b(x,t)dz \]

• Discrete time equivalent
  \[ \Delta x = a(x,t)\Delta t + b(x,t)\varepsilon \sqrt{\Delta t} \]
  • only true in the limit as \( \Delta t \) tends to zero
Ito Processes 2

• Ito Process for Stock Prices

$$\frac{dS}{S} = \mu dt + \sigma dz$$

• where $\mu$ is the expected return
• $\sigma$ is the volatility.

• Discrete time equivalent

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta \epsilon \sqrt{\Delta t}$$

Ito’s Lemma

• Key result
  • If we know the stochastic process followed by $x$,
  • Ito’s lemma tells us
  • the stochastic process followed by a function $G(x, t)$

• Why do we care?
  • a derivative security is a function
    » of the price of the underlying
    » & of time
  • thus, Ito’s lemma plays an important part
    » in the analysis of derivatives
Ito’s Lemma 2

Suppose
\[ dx = a(x,t)dt + b(x,t)dz \]

We have
\[ dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz \]

For a general function \( G(x,t) \). This is Ito’s lemma.

Ito’s Lemma 3

• Application to a Stock Price Process

The stock price process is
\[ dS = \mu S \, dt + \sigma S \, dz \]

For a function \( G \) of \( S \) and \( t \)
\[ dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz \]
Ito’s Lemma 4

• Examples

1. forward price of a stock
   for a contract maturing at time $T$
   \[ G = S \ e^{(T-t)} \]
   \[ dG = (\mu - r)G \ dt + \sigma G \ dz \]

2. $G = \ln S$
   \[ dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz \]

Black-Scholes Analysis

• Lognormal distribution
  • expected return
  • variance

• Markov Processes
  • market efficiency

• Wiener Processes

• Ito’s Lemma
The Lognormal Property

• Definition
  » \( \ln S \) follows a generalized Wiener process with drift \( \mu - \sigma^2/2 \)

• Implications
  \[
  \ln S_T - \ln S = \phi \left( \mu - \frac{\sigma^2}{2}, \sqrt{T-t}, \sigma \sqrt{T-t} \right) \quad (13.2) \quad \text{p. 278}
  \]
  \[
  \ln S_T = \phi \left( \ln S + \left( \mu - \frac{\sigma^2}{2} \right)(T-t), \sigma \sqrt{T-t} \right) \quad (13.3) \quad \text{p. 278}
  \]

• Put differently
  • \( S_T \) is lognormally distributed
  • since the logarithm of \( S_T \) is normal

The Lognormal Property 2

• Lognormal Distribution

\[
E(S_T) = S e^{\mu(T-t)}
\]
\[
\text{var}(S_T) = S^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1)
\]
The Lognormal Property 3

- Continuously Compounded Return, \( \eta \)
  \[ S_T = S \, e^{\eta(T-t)} \]
  or
  \[ \eta = \frac{1}{T-t} \ln \frac{S_T}{S} \]
  \[ \eta = \phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T-t}} \right) \]

The Lognormal Property 4

- The Expected Return
  - Two possible definitions:
    - \( \mu \)
      » is the arithmetic average
      » of the returns realized
      » in many short intervals of time
    - \( \mu - \sigma^2/2 \)
      » is a geometric average
      » is the expected continuously compounded return
      » realized over any finite period of time
The Lognormal Property 5

• The Volatility
  – definition
    • The volatility, \( \sigma \) is the standard deviation
    • of the continuously compounded rate of return
    • in 1 year
  – approximation
    • standard deviation of the proportional change
    • in 1 year

The Lognormal Property 6

• Estimating Volatility from Historical Data
  1. Take observations \( S_0, S_1, \ldots, S_n \)
     at intervals of \( \tau \) years
  2. Define the continuously compounded return as: \( u_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \)
  3. Calculate the standard deviation (\( \sigma \)) of the \( u_i \)'s
  4. Calculate the annualized (historical) volatility: \( s^* = \frac{s}{\sqrt{\tau}} \)
Concepts Underlying Black-Scholes

- Assume constant risk-free interest rate
- Risk-neutral valuation
  - Option price & stock price
    - depend on the same underlying source of uncertainty, $dz$
  - Investor can form a portfolio
    - consisting of the stock & the option
    - which eliminates this source of uncertainty
  - That portfolio is *instantaneously* riskless
    - & thus must *instantaneously* earn the risk-free rate
  - This leads to the B&S differential equation

Derivation of the Black-Scholes Differential Equation

\[
\Delta S = \mu S \Delta t + \sigma S \Delta z
\]

\[
\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z
\]

We set up a portfolio consisting of
- $-1$: derivative
- $+ \frac{\partial f}{\partial S}$: shares
Derivation of the Black-Scholes Differential Equation 2

The value of the portfolio $\Pi$ is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time $\Delta t$ is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

Derivation of the Black-Scholes Differential Equation 3

The return on the portfolio must be the risk-free rate.

Hence $\Delta \Pi = r \Pi \Delta t$

We substitute for $\Delta f \& \Delta S$ in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$
The B&S Differential Equation

• Any security
  • whose price is dependent on the stock price
  • satisfies the differential equation

• The particular security being valued
  • is determined by the boundary conditions
  • of the differential equation
  • e.g. \( \text{Max}(S-X,0) \) for \( t=T \).

Example: Forward Contract

• Boundary condition
  • \( f = S - K \) when \( t = T \)

• Solution to the equation
  \[ f = S - K e^{-r(T-t)} \]

• It is not usually that easy to solve!
  • Risk-Neutral Valuation makes life easier.
Risk-Neutral Valuation

- The variable $\mu$
  - does *NOT* appear in the Black-Scholes equation
- The B&S equation
  - is *independent* of investors’ *risk preferences*
- The *solution* to the differential equation
  - is therefore the *same* in a risk-free world
  - as it is in the *real world*
- Leads to *principle* of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the *expected return*
   from the *stock price*
   is the *risk-free rate*
2. Calculate the *expected payoff*
   from the *option*
3. Discount at the *risk-free rate,*
Example: R-N Value of Forward

- Payoff from Forward: $S_T - K$
- R-N forward value: $f = e^{-r(T-t)} \hat{E}(S_T - K)$
- R-N Stock: $\hat{E}(S_T) = S e^{r(T-t)}$
- So, $f = S - K e^{-r(T-t)}$
- Consistent with Chapter 5 (H7) & Chapter 3 (H6)!

Example: R-N Value of Call Option

- For a European call option,
  
  $$c = e^{-r(T-t)} \hat{E} \left[ \max(S_T - X, 0) \right]$$

- In a risk-neutral world
  
  - $S_T$ will earn the risk-free rate on average, so
  
  $$\ln(S_T) = \phi [\ln(S) + (r - \sigma^2 / 2) (T - t), \sigma \sqrt{T - t}]$$

- Solving the expectations integral, you get $\hat{E}[*]$
Black-Scholes formula

• Black-Scholes
  • gives price of European call
  
  $$c = e^{-r(T-t)}[S N(d_1) e^{r(T-t)} - X N(d_2)]$$

  where

  $$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

  $$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}$$

  • interpretation?

Black-Scholes formula 2

$$c = e^{-r(T-t)}[S N(d_1) e^{r(T-t)} - X N(d_2)]$$

• $N(z) = \text{Prob}(Z<z)$
  » $Z$ is standard normal

• $N(d_2)$
  » = probability of exercise.

• $XN(d_2)$
  » = expected pay-out at exercise

• $SN(d_1) \exp(r(T-t))$
  » = expected value of the stock price, if exercised.
Black-Scholes Properties

• When $S$ gets very large,
  • $N(d_1) = N(d_2) = 1$, and
  \[ c = S - X e^{-r(T-t)} = \text{value of forward} \]

• When $\sigma$ gets very small, again
  • $N(d_1) = N(d_2) = 1$, and $c = S - X e^{-r(T-t)}$

European Put Options

• Just use put-call parity. You’ll get,
  \[ p = e^{-r(T-t)} [ X N(-d_2) - S N(-d_1) e^{r(T-t)} ] \]

• The normal distribution is symmetric, so
  $N(-d_1) = [1-N(d_1)]$
Example 13.6 (12.7 in H6): B-S Pricing

- The stock price today is $42. There is 6 months to expiration. The exercise price is $40. The risk-free rate is 10% and the volatility is 20% per year.
- What is the price of a European call?
- What is the price of a European put?
- Get $N(d_1)$ and $N(d_2)$ from pp. 800-801 (H7).

Warrants & Dilution

- When a regular call option is exercised the stock that is delivered must be purchased in the open market.
- When a warrant is exercised new treasury stock is issued by the company.
- This will dilute the value of the existing stock.
- One valuation approach is to assume that all equity (warrants + stock) follows a geometric Brownian motion, and then use Black-Scholes.
Implied Volatility

- Black-Scholes: $c = BS(S,X,r,T-t,\sigma)$
- Only $\sigma$ is unobserved and must be estimated.
- Call price is observed in market, thus we can back out the so-called B-S implied volatility, theoretically: $\sigma_{iv} = BS^{-1}(S,X,r,T-t,c)$
- No closed-form solution available, but it can be calculated by trial-and-error.

Implied Volatility (cont.)

- There’s a 1-to-1 relationship between the B-S price and IV, and it can be used to estimate the price of one option from another.
- IV’s are used to monitor the market’s view on volatility, as it changes over time.
- Composite IV’s are calculated for a given stock using options with different $X$ and $T-t$. 
Measuring Volatility

• Volatility is much larger on trading days than on non-trading days.
• Black-Scholes is sometimes adjusted using 252 (trading) days per year for volatility and 365 days per year for risk-free interest rate.
• The difference is only significant for short time-to-maturity options.

Dividends

• View dividends as a riskless component of the stock price.
• Use Black-Scholes, but set 
  \[ S = \text{Today’s stock price - PV of dividends} \]
• Example 13.10 (H7): Suppose \( S = X = 40, r = 9\%, \sigma = 30\% \), $0.50 dividends in 2 and 5 months, \( T-t = 6 \) months. What is the price of a European call?
American Calls

- An American call on a NON-dividend-paying stock should NEVER be exercised early, so $C = c$.
- An American call on a dividend-paying stock should ONLY ever be exercised immediately prior to an ex-dividend date.

Black’s Approach to Dealing with Dividends in American Call Options

Set the American price = to the MAX(of 2 European prices):
1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date
The approximation works well since the dividend yield is usually lower than the risk-free rate. It pays to wait until that last ex-dividend date.
Black’s Approach (cont.)

- Example 13.10 (H7): Suppose $S = X = 40$, $r = 9\%$, $\sigma = 30\%$, $0.50$ dividends in 2 and 5 months, $T-t = 6$ months. What is the price of an American call?
- The exact RGW formula in Appendix 12.A gives a slightly different result. It allows for the investor to decide on exercise at the ex-dividend date. You are not responsible for it.

Options on Stock Indices, Currencies and Futures Contracts

Chapter 14
Recap 1: Black-Scholes

- In Black-Scholes, the underlying asset pays no income during the life of the option

\[
c = e^{-r(T-t)} \left[ S \, N(d_1) e^{(r-t)} - X \, N(d_2) \right]
\]

where

\[
d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}
\]

Recap 2: Discrete Dividends

- We can easily adjust Black-Scholes
  - to account for discrete dividend payments
  - by subtracting from the stock a risk-free component equal to the PV of the future dividends.
European Options on Stocks Paying Continuous Dividends

We get the same probability distribution for the stock price at time $T$ in each of the following cases:

1. The stock starts at price $S$ & provides a continuous dividend yield $= q$
2. The stock starts at price $S e^{-q(T-t)}$ & provides no income

Stock Price Distributions

- Recall the distribution of log-stock prices
  \[ \ln(S_T) = N(\ln(S) + (\mu - \sigma^2 / 2)(T-t), \sigma\sqrt{T-t}) \]

- If continuous dividend yield is paid
  \[ \ln(S_T) = N(\ln(S) + (\mu - q - \sigma^2 / 2)(T-t), \sigma\sqrt{T-t}) \]

- Equivalently, if initial stock price is $Se^{-q(T-t)}$
  \[ \ln(S_T) = N(\ln(Se^{-q(T-t)}) + (\mu - \sigma^2 / 2)(T-t), \sigma\sqrt{T-t}) \]
Trick to Extend Black-Scholes

- When pricing derivatives on an underlying asset with continuous dividend yield, $q$,
- we can value European options
- by reducing the stock price to $S e^{-q(T-t)}$ & then behaving as though there is NO dividend

European Options on Stocks
Paying Continuous Dividends

\[ c = S e^{-q(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2) \]
where
\[ d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\ln(S/X) + (r - q - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]
Bounds for Option Prices, w/ q>0

- European call prices w/ continuous dividends using Chapter 7 formula
  \[ c > S e^{-q(T-t)} - X e^{-r(T-t)} \]

- European put prices w/ continuous dividends
  \[ p > X e^{-r(T-t)} - S e^{-q(T-t)} \]

Call-Put Parity, w/ q > 0

- Again, just substitute \( S e^{-q(T-t)} \) for \( S \) in the Chapter 9 PCP formula to get
  \[ p + S e^{-q(T-t)} = c + X e^{-r(T-t)} \]

- Use call-put parity (PCP) to find put option value
  \[ p = X e^{-r(T-t)} N(-d_2) - S e^{-q(T-t)} N(-d_1) , \]
Index Options

- Option contracts are on $100 \times$ the index
- The most popular underlying indices are the S&P 100 (OEX), the S&P 500 (SPX), & the Major Market Index (XMI)
- S&P 100 and XMI options are American, S&P 500 options are European.
- Continuous dividend framework useful for valuation of European index options.

Index Option Example

- Consider a call option on an index with a strike price of 560
- Suppose 1 contract is exercised when the index level is 580
- What is the payoff?
Using **Index Options** for Portfolio Insurance

- Suppose the value of the index is $S$ & the strike price is $X$
- If a portfolio has a $\beta$ of 1.0, the portfolio manager *buys* 1 put option for each 100$S$ dollars held
- If the $\beta$ is *not* 1.0, the portfolio manager *buys* $\beta$ put options for each 100$S$ dollars held
- In both cases, $X$ is chosen to give the appropriate insurance level

Example: Portfolio Insurance w/ $\beta=1$

- The value of a well diversified stock portfolio is $500,000. The index is at 250 points. Suppose the desired insurance value is $480,000 in three months.
- Insurance: Buy 20 put options with $X=240$.
- If index falls to 225, the stocks will be worth about $450,000, but the options will be worth $20*(240-225)*100 = $30,000, thus keeping the insurance value intact.
Example: Portfolio Insurance w/ $\beta \neq 1$

- First find relationship between end-of-period index value and end-of-period portfolio value.
- Then pick strike price for index options corresponding to the desired insurance value of the portfolio.

Valuing European Index Options

We can use the formula for an option on a stock paying a continuous dividend yield

- Set $S =$ current index level
- Set $q =$ average dividend yield per year expected during the life of the option
- Be careful with seasonality in dividend payments.
Example 15.1 (H7)

- What is the value of a European call option on the S&P500 with two months to maturity, when the current value of the index is 930, the strike price is 900, the risk-free rate is 8%, volatility is 20%, and dividend yields of 0.2% and 0.3% are expected in the first and second month respectively?
- **Answer**: $T = 2/12 = 0.167; \, q = \text{dividend yield} = 0.5\% (=0.2\% + 0.3\%); \, d_1 = 0.544; \, d_2 = 0.4328; \, N(d_1) = 0.7069; \, N(d_2) = 0.6782; \, c = 51.83 \rightarrow \text{contract cost} = 5183$

Currency Options

- **Currency options** trade on the Philadelphia Exchange (PHLX)
- They are used by corporations to **hedge** their FX exposure
- The size of a contract **depends** on the currency
  - Examples: 62,500 GBP, 125,000 CHF
- There also exists an active over-the-counter (OTC) market
**Valuing European Currency Options**

- A **foreign currency** is an instrument that provides a continuous income equal to the **foreign risk-free rate** ($r_f$).
- We can **use** the formula for an option on a stock paying a **continuous dividend yield**:
  - Set $S = \text{current exchange rate}$
  - Set $q = r_f$
- Note that the income earned in the **domestic currency** is $r_f S$ showing that $r_f$ is **analogous** to $q$.

**European Currency Options**

- **Pricing formulas**

  $$c = S e^{-r_f(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2)$$
  
  where

  $$d_1 = \frac{\ln(S/X) + (r - r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

  $$d_2 = \frac{\ln(S/X) + (r - r_f - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$
Put-Call Parity for Currency Options

• Again, just substitute for $S$ in Chapter 7 formula to get new parity
  \[ Se^{-r(T-t)} + p = c + Xe^{-r(T-t)} \]
• Use put-call parity to find put price
  \[ p = X \ e^{-r(T-t)} \ N(-d_2) - S \ e^{-r(T-t)} \ N(-d_1), \]

Simplification using Forward Price

• Recall we have \( F = Se^{(r-r_c)(T-t)} \)
• Thus we can simplify Black-Scholes to write \( c = e^{-r(T-t)}[FN(d_1) - XN(d_2)] \)
  where
  \[ d_1 = \frac{\ln(F/X) + (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
  \[ d_2 = \frac{\ln(F/X) - (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]
Futures Options (Chapter 16)

• When a **CALL** is **exercised**, the holder acquires a **LONG** position in the **futures** plus a **cash** amount equal to the excess of
  - the **futures price** over
  - the **strike price**

• When a **PUT** is **exercised** the holder acquires a **SHORT** position in the **futures** plus a **cash** amount equal to the excess of
  - the **strike price** over
  - the **futures price**

Futures Options [cont.]

• Underlying is a futures contract which expires a few days after the expiration of the option.

• Very popular due to
  - Cheaper delivery (hogs vs. hog futures)
  - Often settled in cash (futures position closed out)
  - Traded side-by-side with underlying
Example

• An investor has a September futures call option on 25,000 pounds of copper with a strike price of 70 cents per pound. The current futures price of September copper is 80 cents. If exercised, the investors receives $2,500 plus a long futures position in 25,000 pound of September copper. The futures position can be closed out immediately at no cost.

Valuing European Futures Options

• We can use the formula for an option on a stock which pays a continuous dividend yield:
  - Set $S = \text{current futures price (} F \text{)}$
  - Set $q = \text{domestic risk-free rate (} r \text{)}$
• Setting $q = r$ ensures that the expected growth of $F$ in a risk-neutral world is $\mathcal{O}$
Black's Formula

- The formulas for European options on futures are sometimes referred to as Black's formulas

\[ c = e^{r(T-t)} \left[ F N(d_1) - X N(d_2) \right] \] (16.9) p.350

\[ p = e^{r(T-t)} \left[ X N(-d_2) - F N(-d_1) \right] \] (16.10) p.350

where

\[ d_1 = \frac{\ln(F/X) + \sigma^2(T-t)/2}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\ln(F/X) - \sigma^2(T-t)/2}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

Growth Rates For Futures Prices

- A futures contract requires NO initial investment
- In a risk-neutral world the expected return should be Ø
- The expected growth rate of the futures price is therefore Ø
- The futures price can therefore be treated like a stock paying a dividend yield of \( r \)
Summary of Put-Call Parities

Indices:
\[ c + X e^{-r(T-t)} = p + S e^{q(T-t)} \]

Foreign exchange:
\[ c + X e^{-r(T-t)} = p + S e^{-r_f(T-t)} \]

Futures:
\[ c + X e^{-r(T-t)} = p + F e^{-r(T-t)} \quad (12.13) \text{ p.278} \]

Summary of other Key Results

- We can treat stock indices, currencies, & futures like a stock paying a continuous dividend yield of \( q \)
  - For stock indices, \( q = \text{average dividend yield per year} \) on the index over the option life
  - For currencies, \( q = r_f \)
  - For futures, \( q = r \)