Derivatives & Risk Management

• First Week:
  – Part A: Option Fundamentals
    • payoffs
    • market microstructure
• Next 2 Weeks:
  – Part B: Option Pricing
    • fundamentals: intrinsic vs. time value, put-call parity
    • introduction to the Black-Scholes pricing model
    • binomial trees & risk-neutral valuation

Part V:
Option Pricing Basics

Option Pricing Principles

• Fundamentals
  • time value vs. intrinsic value
  • key determinants of option values
  • American vs. European options – Early exercise
• Put-call parity
  • non-dividend paying stocks
  • dividend adjustment
• Option pricing
  • Black-Scholes formula
Option Pricing Principles: Notation

- $X$: Strike price = exercise price
- $c$: European call option price
- $p$: European put option price
- $C$: American call option price
- $P$: American put option price
- $t$: Current time
- $T$: Maturity = time when option expires
- $S_t$: Spot price at time $t$
- $\sigma$: Volatility of the underlying’s price
- $D$: PV of Dividends
- $r$: Relevant risk-free rate (continuous compounding)

Option Pricing Principles 2

- intrinsic value vs. time value
  - intrinsic value
    - calls: $\text{Max}(0, S_t - X)$
    - put: $\text{Max}(0, X - S_t)$
  - time value = option premium minus intrinsic value
    - at worst, equal to 0 (note: European vs. American)
    - strictly positive for out-of-the-money options
    - usually positive for in-the-money options

Option Pricing Principles 3

- Key determinants of option prices
  - American options vs. European options
    - at least as valuable ($C \geq c, P \geq p$)
    - equal values at maturity
  - time to maturity
    - American options: $T \uparrow \Rightarrow \ P \uparrow$ and $C \uparrow$
    - European options?
  - strike price
    - $X \uparrow \Rightarrow \ p \uparrow$ but $c \downarrow$
Option Pricing Principles 4

- Key determinants of option prices (continued)
  - price of underlying asset
    - \( S_t \uparrow \Rightarrow p & P \downarrow \) but \( c & C \uparrow \)
  - Dividends
    - \( D \uparrow \Rightarrow c & C \downarrow \) but \( p & P \uparrow \)
    - IV (European options) vs. TV effect (American options)
  - volatility of underlying asset
    - \( \sigma \uparrow \Rightarrow p & P \uparrow \) and \( c & C \uparrow \) (intuition?)
  - hard floors vs. soft floors

Option Pricing Principles 5

- H7 Table 9.1; H8 Table 10.1

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Option Pricing Principles 6

- Hard and Soft Floors
  - hard floor (American calls)
    - \( C_t = \text{Max}[0, S_t - X] \)
    - if not satisfied, arbitrage exists (buy call & strike now)
  - soft floor (all calls, but only on non-dividend paying stocks)
    - \( S_t \leq c_t \Rightarrow C_t = \text{Max}[0, S_t - X/(1+r)^T] \)
    - if not satisfied, arbitrage exists (buy call & risk-free bond)
    - consequence: early exercise of American calls is not optimal if the underlying asset pays no dividends
Option Pricing Principles 7

- Early exercise (American calls)
  - non-dividend paying stocks
    » never optimal to exercise early
    » intuition:
      \[ C_t \geq \max[0, S_t - X / (1+r)^T] > \max[0, S_t - X] \]
    » corollary: same bound for European calls on such assets
  - dividend paying stocks?
    » early exercise may be optimal…
    » … but only if stock pays large dividend prior to maturity

Option Pricing Principles 8

- Hard and Soft Floors (continued)

**Question:**
Suppose an American call option is written on Nortel stock. The exercise price is $105 (\(J\)) and the present value of the exercise price is $100.

(a) What is the hard floor price of the option if Nortel stock sells for $160? Sketch a graph of the hard floor option prices against (i.e., in terms of) the Nortel stock’s price.
(b) At a stock price of $125, you notice the option selling for $18. Would this option price be an equilibrium price? Explain.

Answer:

(a) Hard floor price = \( V_t - X \)
\[ = 160 - 105 = 55 \]

(b) An option price of $18 is below the hard floor price of $20. In this case, everyone would want the call option. You could then acquire a share of Nortel stock for less than the current market price. Simply buy the option (for $18), exercise it (paying $105), and you would then own a share of Nortel for a total price of $123.
Option Pricing Principles 10

• Hard and Soft Floors (American puts)
  • hard floor
    » \( \text{Max}[0, X - S_t] \)
    » if not satisfied, arbitrage exists
  • soft floor?
    » \( \text{Max}[0, X/(1+r)^T - S_t] \)
    » BKM4 Fig. 21.4

Option Pricing Principles 11

• Early exercise (American puts)
  • can be optimal to exercise early
    » intuition 1: stock price cannot fall below 0
    » intuition 2: \( T \uparrow \Rightarrow X/(1+r)^T \downarrow \)
  • impact of dividend payments
    » dividends \( \uparrow \Rightarrow \) probability of early exercise \( \downarrow \)

Options: Early Exercise (Recap)

• Calls
  • often not optimal
    » never optimal for non-dividend paying stocks
    • importance of capturing dividends
  • Puts
    • can be optimal to exercise early
    • impact of dividend payments
      » dividends \( \uparrow \Rightarrow \) probability of early exercise \( \downarrow \)
**Put-Call Parity**

- Put-call parity
  - European options only
  - applicability to American options?
- Intuition
- Formally
  - no-arbitrage
  - if the payoffs of 2 portfolios are equal
  - then the costs of both portfolios must be equal
  - “reverse engineer” the prices
  - examples

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**Put-Call Parity 2**

- Intuition

(a: covered call)

(b) 

(c: protect. put)

(d)

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**Put-Call Parity 3**

- Put-call parity

\[
\begin{array}{ccc}
\text{Cash} & \leq X & \geq X \\
\text{Selling} & S_T - X & 0 \\
\text{Buying} & 0 & X - S_T \\
\text{Selling} & -X & 0 \\
\text{Buying} & S_T & S_T \\
\text{Total} & S_T - X & 0 \\
\end{array}
\]

\[c = p + X/(1+r)^T - S_T \quad \text{and thus} \quad c = p + S_T - X/(1+r)^T\]
**Put-Call Parity 4**

**Question:**
European put and a European call on the same stock
exercise price $X = \$75$
same expiration dates
The current price of the stock is $68.
The put’s current price is $6.50 higher than the call’s price
A riskless investment over the time until expiration yields 3 percent.
Given this information, is there any riskless profit opportunities available?

**Put-Call Parity 5**

**Answer:**
According to the parity equation:

$$V_P - V_C = \frac{X}{(1 + r_f)^t} - V_S \Rightarrow V_P - V_C = \frac{\$75}{(1 + 0.03)^t} - 68 = \$4.82.$$  

Thus, with the put being priced $6.50 higher than the call, the two options are out of parity. A riskless arbitrage opportunity would exist:

**Put-Call Parity 6**

**Answer:**
A riskless arbitrage opportunity exists:

- Sell the stock short...................... $68.00
- Sell the put option......................
- Buy the call option..................... $6.50
- Proceeds: $74.50

Invest the proceeds at the riskless rate of 3%. At maturity, you will have the value at expiration of $76.74 = \$74.50 \times 1.03$.

Also, you can acquire a share of stock (to cover the short sale) for $75, no matter what happens to the stock price.

You are assured $1.74 without putting any of your own money at risk.
Put-Call Parity 7

• Put-call parity (continued)
  • continuous-time version
    » \( c = p + S_t - X e^{-(T-t)} \)
    » \( c - p = S_t - X e^{-(T-t)} \)
  • dividends
    » adjustment needed
    » \( c = p + PV(S_T) - PV(\text{dividend}) - X/(1+r)^T \)

Put-Call Parity 8

• Extensions (NOT Exam Material)
  • American options; \( D = 0 \)
    \[ S - X < C - P < S - X e^{-(T-t)} \] (H8 eq. 10.4)
  • European options; \( D > 0 \)
    \[ c - p = S - D - X e^{-(T-t)} \] (H8 eq. 10.7)
  • American options; \( D > 0 \)
    \[ S - D - X < C - P < S - X e^{-(T-t)} \] (H7, 9.8 p. 215) (H8 eq. 10.11)

Option Pricing Methods

• Analytical
  • Black-Scholes
    » pluses (quick) & minuses (European calls, assumptions)
  • Numerical
    • Binomial Trees
    • Monte Carlo Methods
    • Finite difference Methods
    • Analytical Approximation
Option Pricing – Key Problem

• Uncertainty
  • we don’t know future stock prices

• Solution
  • Assume a distribution for periodic returns
  • Assume a stochastic process for stock prices

Option Pricing in Practice

• Black-Scholes
  • gives price of European call
  \[ c = e^{-rT} [S N(d_1)e^{r(T-t)} - X N(d_2)] \]
  where
  \[ d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
  \[ d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
  • interpretation?

Option Pricing in Practice 2

\[ c = e^{-r(T-t)} [S N(d_1)e^{r(T-t)} - X N(d_2)] \]

• \( N(z) \) = Prob(\(Z<z\))
  \( Z \) is standard normal
• \( N(d_2) \)
  \( = \) probability of exercise.
• \( XN(d_2) \)
  \( = \) expected pay-out at exercise
• \( SN(d_1 \exp(r(T-t))) \)
  \( = \) expected value of the stock price, if exercised.
Option Pricing in Practice 3

- Black-Scholes (continued)
  - gives price of European call
  - price of European put?
    - use put-call parity
    - intuition:

- American options?
  - optimality of early exercise

Option Pricing in Practice 4

- Binomial option pricing
  - problem
    - assumptions needed for B&S are not always realistic
    - example: interest rates are deterministic?!
  - solutions:
    - 1. numerical approximations
      - grid, risk-neutral probabilities
      - as accurate as needed/desired
    - 2. PBS = Practitioners’ B&S (e.g., Berkowitz)

Numerical Pricing Methods

- Risk-Neutral valuation
- Methods
  - Binomial Trees
    - Early Exercise Possible
  - Monte Carlo Methods
    - Several Underlying Variables Possible
  - Finite difference Methods
    - Early Exercise Possible
  - Analytical Approximation
    - American Options
Risk-Neutral Valuation

• Approach
  • introduce binomial trees now
  • to start thinking about
    – risk-neutral valuation of derivatives
    – and dynamic hedging strategies

• Applicability
  • use risk-neutral valuation throughout the course
  • return to binomial trees in Parts III & IV

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Example

• Call Option example (H7 Fig. 11.1; H8 12.1):
  • 3-month call option with strike price X = 21
    
    \[
    \begin{align*}
    \text{Stock Price} &= $22 \\
    \text{Call Price} &= $1 \\
    \text{Stock Price} &= $18 \\
    \text{Call Price} &= $0
    \end{align*}
    \]

  • price of the call today?
    – use risk-neutral valuation

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Example 2

• Riskless Portfolio
  • Portfolio
    \[
    \begin{align*}
    \text{LONG} & \quad \Delta \text{shares} \\
    \text{SHORT} & \quad 1 \text{ call option}
    \end{align*}
    \]
    
    \[
    \begin{align*}
    S &= $20 \\
    \Delta S &= S$22 - $1 = $21 \\
    $18 = S$$18 - $0
    \end{align*}
    \]

  • Portfolio is riskless
    \[
    \begin{align*}
    \text{if} & \quad S22 - $1 = S18 \\
    \text{i.e.} & \quad \Delta = 0.25 \\
    \text{if} & \quad \text{LONG 0.25 shares and SHORT 1 call option}
    \end{align*}
    \]
Example 3

• Value of the riskless portfolio
  – in 3 months
    • if the stock price moves up:
      \[ \times 22 \times 0.25 - 1 = 4.50 \]
    • if the stock price moves down:
      \[ \times 18 \times 0.25 - 0 = 4.50 \]
  – today
    • PV of \$4.50 at the risk-free rate (why?)
    • if annual continuously-compounded risk-free rate is 12%, portfolio is worth: \[ \text{\$4.50 e}^{-0.12 \times 0.25} = 4.367 \]

Example 4

• Value of the Option Today
  • entire portfolio
    » worth \$4.367
  • shares
    » worth \( \Delta \times S = 0.25 \times \$20 = \$5 \)
  • Value of the option
    » is therefore: \( \$5 - 4.367 = 0.633 \)

Binomial Option Pricing Fundamentals

• Why?
  • approximate the movements in an asset’s price
  • to simplify the pricing of derivatives on the asset

• What?
  • “discretize” underlying asset’s price movements
  • and value options as if in a risk-neutral world

• How?
  • asset price at the BEGINNING of any period can lead to
  • only 2 stock prices at the END of that period
Binomial Trees

- Asset Price Movements
  - divide the time from $t$ to $T$ into small intervals $\Delta t$
  - in each time interval, assume the asset’s price $S$ can move $UP \uparrow$
    - by a proportional amount $u$
    - or
  - move $DOWN \downarrow$
    - by a proportional amount $d$

Binomial Trees 2

- Movements in time interval $\Delta t$
  - (H7 Fig.19.1; H8 12.2)
  \[ S \xrightarrow{p} Su, \quad S \xrightarrow{(1-p)} Sd \]

Tree Parameters

- What?
  - $p$, $u$ and $d$
- Parameter values?
  - tree must give correct values
  - for the mean & standard deviation
  - of the stock price changes
  - in a risk-neutral world
- Simplification
  - tree is recombining: $u = 1/d$
Tree Parameters 2

- Complete Tree (Fig. 19.2)

Risk-Neutral Valuation

- Assumption
  - no arbitrage opportunity exists

- Basic idea
  - assume a binomial tree for asset price movements
  - create a riskless portfolio
    - stock plus option
  - riskless portfolio always possible with binomial tree
  - value the portfolio
    - if riskless, then risk-neutral valuation is OK

- Reference

Risk-Neutral Valuation 2

- European Put example \((u=1.1; \ d=1/u)\):
  - 3-month put with strike price \(X = 21\)
    - Stock Price = $22
      - Put Price = $0
    - Stock Price = $20
      - Put Price = ?
    - Stock Price = $18.18
      - Put Price = $2.82
  - price of the put?
    - use risk-neutral valuation
Risk-Neutral Valuation 3
• Riskless Portfolio
  • Portfolio
    - LONG $\Delta$ shares
    - LONG 1 put option
  $S = \Delta = 22$
  \[ \Delta \times 0.738 + 0 = 16.24 \]
  \[ 18.18 \times 0.738 + 2.82 = 13.42 + 2.82 = 16.24 \]
  \[ \text{Portfolio is riskless} \]
  - if \( 22\Delta = 18.18\Delta + 2.82 \) \( \Rightarrow \Delta = 0.738 \)
  - LONG 0.738 shares and LONG 1 put option

Risk-Neutral Valuation 4
• Value of the entire (riskless) portfolio
  - in 3 months
    • if the stock price moves up:
      \[ 22 \times 0.738 + 0 = 16.24 \]
    • if the stock price moves down:
      \[ 18.18 \times 0.738 + 2.82 = 13.42 + 2.82 = 16.24 \]
  - today
    • PV of $16.24$ at the risk-free rate (why?)
    • if annual continuously-compounded risk-free rate is 12%, portfolio is worth: $16.24 e^{-0.12 \times 0.25} = 15.76$

Risk-Neutral Valuation 5
• Value of the Option Today
  • Entire portfolio
    \[ \text{is worth } 15.76 \]
  • Shares
    \[ \text{are worth } 0.738 \times 20 = 14.76 \]
  • Value of the put option
    \[ \text{is therefore } 15.76 - 14.76 = 1.00 \]
Risk-Neutral Valuation 6

- Generalization (*H7 Fig. 11.2; H8 Fig. 12.2*)
  - derivative
    - value \( f \)
    - expires at time \( T \)
    - dependent on a stock

Risk-Neutral Valuation 7

- Riskless portfolio
  - LONG \( \Delta \) shares and SHORT 1 derivative

Risk-Neutral Valuation 8

- Value of the portfolio at time \( T \):
  - \( \text{up state: } S_u \Delta - f_u = S_d \Delta - f_d \) (down state)
- Value of the portfolio today:
  - \( (S_u \Delta - f_u)e^{-rT} \)
  - and also
  - \( S \Delta - f \)
- Hence
  - \( f = S \Delta - (S_u \Delta - f_u)e^{-rT} \)
Risk-Neutral Valuation 9

- Thus:
  \[ f = S \Delta - (Su - f_u) e^{-rT} \]

- Substituting for \( \Delta \), we obtain
  \[ f = \left[ pf_u + (1-p)f_d \right] e^{-rT} \]
  where \( p = \frac{e^{rT} - d}{u - d} \)

Risk-Neutral Valuation 10

- Interpretation
  - \( f = \left[ pf_u + (1-p)f_d \right] e^{-rT} \)
  - \( p \) and \((1-p)\) can be interpreted as the risk-neutral probabilities of up & down movements

- Value of a derivative
  - \( f \): its expected payoff
  - in a risk-neutral world
  - discounted
  - at the risk-free rate

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Irrelevance of Stock’s Expected Return

- When valuing an option in terms of the underlying stock,
  - the expected return on the stock is irrelevant
Original Example Revisited

• Call, H8 Fig. 12.1 (\(S = 20; X = 21; \Delta t = T = 3\) months)

\[
\begin{align*}
S_u &= 22 \\
S_d &= 18 \\
\Delta t &= T = 3 \text{ months}
\end{align*}
\]

* risk-neutral probabilities:

\[
p = \frac{e^{\Delta t} - d}{u - d} = \frac{e^{0.12\times0.25} - 0.9}{1.1 - 0.9} = 0.6523
\]

Original Example Revisited 2

• H8 Fig. 12.1

\[
\begin{align*}
S_u &= 22 \\
S_d &= 18 \\
f_u &= 1 \\
f_d &= 0
\end{align*}
\]

* Value of the option

\[
c = e^{-0.12\times0.25} \left( 0.6523 \times $1 + 0.3477 \times $0 \right) = $0.633
\]

Original Example Revisited 3

• Key Result
  – risk-neutral valuation ("revisited")
  – coincides with the ("original") no-arbitrage valuation.

• Generalization
  • in general
  • when pricing derivatives
  • using risk-neutral valuation
  • is ok
Original Example Revisited 4

• Valuing the Stock
  • in a risk-neutral world
    – stock must also earn the risk-free rate
  • consequence
    – Since $p$ is a risk-neutral probability
    – $20 \cdot e^{0.12 \cdot 0.25} = 22 \cdot p + 18 \cdot (1 - p)$
    – $p = 0.6523$

A Two-Step Call Option Example

• H8 Fig. 12.3 (X=21; u=1.1; d=0.9; T=6 months)

Each "time step" is $\Delta t=3$ months

A Two-Step Call Option Example 2

• Call value (Fig. 12.4; X=21)

• Value at node B = $e^{-0.12 \cdot 6 \cdot 0.5} (0.6523 \cdot 3.2 + 0.3477 \cdot 0) = 2.0257$
• Value at node A = $e^{-0.12 \cdot 6 \cdot 0.5} (2.0257 + 0.3477 \cdot 0) = 1.2823$
• No difference between American & European calls
A Two-Step Put Option Example

- **Fig. 12.7** (X=52; u=1.2, d=0.8, T=2 years)

Each "time step" is Δt=1 year; annual risk-free rate = r = 5%

\[
p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282
\]

A Two-Step Put Option Example 2

- **European put value**
  (Fig. 12.7; X=52)

- Value at node \( B \) = \( e^{-0.05} (0.6282 \times 50 + 0.3718 \times 4) = 1.4147 \)
- Value at node \( C \) = \( e^{-0.05} (0.6282 \times 4 + 0.3718 \times 20) = 9.4636 \)
- Value at node \( A \) = \( e^{-0.05} (0.6282 \times 1.4147 + 0.3718 \times 9.4636) = 4.1923 \)

A Two-Step Put Option Example 3

- **American put value**
  (Fig. 12.8; X=52)

- Value at node \( B \) = \( \max[X - S_B, 1.4147] = 1.4147 \)
- Value at node \( C \) = \( \max[X - S_C, 9.4636] = \max[12, 9.4636] = 12 \)
- Value at node \( A \) = \( \max[2, e^{0.05} (0.6282 \times 1.4147 + 0.3718 \times 12)] = 5.0894 \)
Delta

- Definition
  - Delta ($\Delta$) is the ratio
    - of the change in the price of a stock option
    - to the change in the price of the underlying stock

- Dynamic hedging
  - The value of $\Delta$ varies from node to node
    - Dynamic hedging needed!

Delta 2
- Riskless portfolio at a given node:
  - LONG $\Delta$ shares and SHORT 1 derivative
    - $\Delta S - f_u$
    - $\Delta S - f_d$
  - riskless
    - if $Su \Delta - f_u = Sd \Delta - f_d$

Delta 3
- European put ($S=50; X=52$)
  - Value of $\Delta$ at node B
    - $=(0-4)/(72-48) = -1/6$ (i.e., short 1 put and short 1/6 share)
  - Value at node C
    - $=-(4-20)/(48-32) = -1$ (i.e., short 1 put and short 1 share)
  - Value at node A
    - $=-(1.41-9.46)/(60-40) = -0.4025$ (short 1 put and short 0.4025 share)