Derivatives & Risk Management

First Week:
- Part A: Option Fundamentals
  - payoffs
  - market microstructure

Next 2 Weeks:
- Part B: Option Pricing
  - fundamentals: intrinsic vs. time value, put-call parity
  - introduction to the Black-Scholes pricing model
  - binomial trees & risk-neutral valuation

Part V:
Option Pricing Basics

Option Pricing Principles

- Fundamentals
  - time value vs. intrinsic value
  - key determinants of option values
  - American vs. European options – Early exercise

- Put-call parity
  - non-dividend paying stocks
  - dividend adjustment

- Option pricing
  - Black-Scholes formula

Option Pricing Principles: Notation

- $X$: Strike price = exercise price
- $c$: European call option price
- $p$: European put option price
- $C$: American call option price
- $P$: American put option price
- $t$: Current time
- $T$: Maturity = time when option expires
- $S_t$: Spot price at time $t$
- $\sigma$: Volatility of the underlying’s price
- $D$: PV of Dividends
- $r$: Relevant risk-free rate (continuous compounding)

Option Pricing Principles 2

- intrinsic value vs. time value
  - intrinsic value
    - calls: Max(0, $S_t - X$)
    - put: Max(0, $X - S_t$)
  - $t$ value = option premium minus intrinsic value
    - at worst, equal to $0^+$ (nope: European vs. American)
    - strictly positive for out-of-the-money options
    - usually positive for in-the-money options

Option Pricing Principles 3

- Key determinants of option prices
  - American options vs. European options
    - at least as valuable
    - equal values at maturity
  - time to maturity
    - American options: $T \uparrow \Rightarrow P \uparrow$ and $C \uparrow$
    - European options?
  - strike price
    - $X \uparrow \Rightarrow P \uparrow$ but $C \downarrow$
**Option Pricing Principles 4**

- Key determinants of option prices (continued)
  - price of underlying asset
    - $S_t \uparrow \Rightarrow p&P \downarrow$ but $c&C \uparrow$
  - Dividends
    - $D \uparrow \Rightarrow c&C \downarrow$ but $p&P \uparrow$
  - IV (European options) vs. TV effect (American options)
  - volatility of underlying asset
    - $\sigma \uparrow \Rightarrow p&P \uparrow$ and $c&C \uparrow$ (intuition?)
  - hard floors vs. soft floors

**Option Pricing Principles 5**

<table>
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<tr>
<th>Variable</th>
<th>$c$</th>
<th>$p$</th>
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<td>$D$</td>
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**Option Pricing Principles 6**

- Hard and Soft Floors
  - hard floor (American calls)
    - $C_t = \text{Max}[0, S_t - X]$  
    - if not satisfied, arbitrage exists (buy call & strike now)
  - soft floor (all calls, but only on non-dividend paying stocks)
    - $S_t \leq C_t = \text{Max}[0, S_t - X/(1+r)^T ]$  
    - if not satisfied, arbitrage exists (buy call & risk-free bond)
    - consequence: early exercise of American calls is not optimal if the underlying asset pays no dividends

**Option Pricing Principles 7**

- Early exercise (American calls)
  - non-dividend paying stocks
    - never optimal to exercise early
      - intuition: $C_t = \text{Max}[0, S_t - X/(1+r)^T ] > \text{Max}[0, S_t - X]$  
      - corollary: same bound for European calls on such assets
  - dividend paying stocks?
    - early exercise may be optimal…
    - … but only if stock pays large dividend prior to maturity

**Option Pricing Principles 8**

- Hard and Soft Floors (continued)

**Question:**
Suppose an American call option is written on Nortel stock. The exercise price is $105 (\$105$) and the present value of the exercise price is $100.

(a) What is the hard floor price of the option if Nortel stock sells for $160? Sketch a graph of the hard floor option prices against (i.e., in terms of) the Nortel stock's price.

(b) At a stock price of $125, you notice the option selling for $18. Would this option price be an equilibrium price? Explain.

**Option Pricing Principles 9**

- Hard and Soft Floors (continued)

**Answer:**
(a) Hard floor price = $V_t - X = $160 - $105 = $55.
(b) An option price of $18 is below the hard floor price of $20. In this case, everyone would want the call option. You could then acquire a share of Nortel stock for less than the current market price. Simply buy the option (for $18), exercise it (paying $105), and you would then own a share of Nortel for a total price of $123.
Option Pricing Principles 10

- Hard and Soft Floors (American puts)
  - Hard floor
    - Max[0, X - S_t]
    - if not satisfied, arbitrage exists
  - Soft floor?
    - Max[0, X / (1+r)^T - S_t]
    - BKM4 Fig. 21.4

Option Pricing Principles 11

- Early exercise (American puts)
  - can be optimal to exercise early
    - intuition 1: stock price cannot fall below 0
    - intuition 2: T ↑ ⇒ X / (1+r)^T ↓
  - impact of dividend payments
    - dividends ↑ ⇒ probability of early exercise ↓

Options: Early Exercise (Recap)

- Calls
  - often not optimal
    - never optimal for non-dividend paying stocks
  - importance of capturing dividends
- Puts
  - can be optimal to exercise early
  - impact of dividend payments
    - dividends ↑ ⇒ probability of early exercise ↓

Put-Call Parity

- Put-call parity
  - European options only
    - if the payoffs of 2 portfolios are equal
  - Intuition
    - “reverse engineer” the prices
    - examples

Put-Call Parity 2

- Intuition

Put-Call Parity 3

- Put-call parity

\[
\begin{array}{c|c|c|c}
\text{ portfolio } & \text{ S_T - X } & \text{ X - S_T } & \text{ cash now} \\
\hline
\text{1. buy a call} & S_T - X & 0 & -c \\
\text{OR} & & & \\
\text{2a. buy a put} & 0 & X - S_T & -p \\
\text{2b. sell disc. bond} & -X & -X & X / (1+r)^T \\
\text{2c. buy stock} & S_T & S_T & -S_0 \\
\text{2. Total} & S_T - X & 0 & \\
\end{array}
\]

- hence: \( c = -p + X / (1+r)^T - S_0 \) and thus \( c = p + S_T - X / (1+r)^T \)
Put-Call Parity 4

Question:
European put and a European call on the same stock
exercise price $X = 75$
same expiration dates

The current price of the stock is $68.
The put’s current price is $6.50 higher than the call’s price
A riskless investment over the time until expiration yields 3 percent.

Given this information, is there any riskless profit opportunities available?

Put-Call Parity 5

Answer:
According to the parity equation:

$V_P - V_C = \left[ \frac{X}{1 + r_f} \right] - V_S = \left[ \frac{75(1 + 0.03)}{1} \right] - 68 = 54.82.$

Thus, with the put being priced $6.50 higher than the call, the two options are out of
parity. A riskless arbitrage opportunity would exist:

Put-Call Parity 6

Answer:
A riskless arbitrage opportunity exists:

Sell the stock short......................... $68.00
Sell the put option......................... $6.50
Buy the call option............... $74.50
Proceeds

Invest the proceeds at the riskless rate of 3%. At maturity,
you will have the value at expiration of $76.74 = \left[ 74.50 \times 1.03 \right].
Also, you can acquire a share of stock (to cover the short sale) for $75, no matter what happens to the stock price.

You are assured $1.74 without putting any of your own money at risk

Put-Call Parity 7

• Put-call parity (continued)

• continuous-time version

$\begin{align*}
& c = p + S_t - X e^{r(T-t)} \\
& c - p = S_t - X e^{r(T-t)} \\
& \text{dividends} \\
& c = p + PV(S_T) - PV(dividend) - \frac{X}{(1+r)^T} 
\end{align*}$

Put-Call Parity 8

• Extensions (NOT Exam Material)

• American options; $D = 0$

$S - X < C - P < S - X e^{r(T-t)}$ (H8 eq. 10.4)

• European options; $D > 0$

$c - p = S - D - X e^{-r(T-t)}$ (H8 eq. 10.7)

• American options; $D > 0$

$S - D - X < C - P < S - X e^{-r(T-t)}$ (H7, 9.8 p. 215)

$S - D - X < C - P < S - X e^{-r(T-t)}$ (H8 eq. 10.11)

Option Pricing Methods

• Analytical

• Black-Scholes

• pluses (quick) & minuses (European calls, assumptions)

• Numerical

• Binomial Trees

• Monte Carlo Methods

• Finite difference Methods

• Analytical Approximation
Option Pricing – Key Problem

• Uncertainty
  • we don’t know future stock prices

• Solution
  • Assume a distribution for periodic returns
  • Assume a stochastic process for stock prices

Option Pricing in Practice

• Black-Scholes
  • gives price of European call
  \[ c = e^{-r(T-t)}[S N(d_1)e^{r(T-t)} - X N(d_2)] \]
  where
  \[ d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \]
  \[ d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \]

• interpretation?

Option Pricing in Practice 2

\[ c = e^{-r(T-t)}[S N(d_1)e^{r(T-t)} - X N(d_2)] \]

• \( N(z) = \text{Prob}(Z<z) \)
  • \( Z \) is standard normal

• \( N(d_1) \)
  • probability of exercise.

• \( XN(d_2) \)
  • expected pay-out at exercise

• \( SN(d_1\exp(r(T-t))) \)
  • expected value of the stock price, if exercised.

Option Pricing in Practice 3

• Black-Scholes (continued)
  • gives price of European call
  • price of European put?
    • use put-call parity
    • intuition:
      • American options?
        • optimality of early exercise

Numerical Pricing Methods

• Risk-Neutral valuation

• Methods
  • Binomial Trees
    • Early Exercise Possible
  • Monte Carlo Methods
    • Several Underlying Variables Possible
  • Finite difference Methods
    • Early Exercise Possible
  • Analytical Approximation
    • American Options
Risk-Neutral Valuation

• **Approach**
  • introduce binomial trees now
  • to start thinking about
    – risk-neutral valuation of derivatives
    – and dynamic hedging strategies

• **Applicability**
  • use risk-neutral valuation throughout the course
  • return to binomial trees in Parts III & IV

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Example

• **Call Option example** *(H7 Fig. 11.1; H8 12.1):*
  • 3-month call option with strike price \( X = 21 \)
  • 3-month call option with strike price \( X = 21 \)

  - Stock Price = \$22
    - Call Price = \$1
  - Stock Price = \$20
    - Call Price = ?
  - Stock Price = \$18
    - Call Price = \$0

  • price of the call today?
    – use risk-neutral valuation

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Example 2

• **Riskless Portfolio**
  • Portfolio
    - _LONG_ \( \Delta \) shares
    - _SHORT_ 1 call option
  • Portfolio is **riskless**
    - if \( 22\Delta - 1 = 18\Delta \), i.e. if \( \Delta = 0.25 \)
    - _LONG_ 0.25 shares and _SHORT_ 1 call option

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Example 3

• **Value of the riskless portfolio**
  - in 3 months
    • if the stock price moves up:
      \( 22 \times 0.25 - 1 = 4.50 \)
    • if the stock price moves down:
      \( 18 \times 0.25 - 0 = 4.50 \)
  - today
    • PV of 4.50 at the risk-free rate (why?)
    • if annual continuously-compounded risk-free rate is 12%, portfolio is worth: \( 4.50 e^{-0.12 \times 0.25} = 4.367 \)

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Example 4

• **Value of the Option Today**
  • entire portfolio
    » worth \$4.367
  • shares
    » worth \( \Delta \times S = 0.25 \times 20 = \$5 \)
  • _Value of the option_
    » is therefore: \( \$5 - 4.367 = \$0.633 \)

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Binomial Option Pricing Fundamentals

• **Why?**
  • approximate the movements in an asset’s price
  • to simplify the pricing of derivatives on the asset

• **What?**
  • “discretize” underlying asset’s price movements
  • _and_ value options as if in a risk-neutral world

• **How?**
  • asset price at the BEGINNING of any period can lead to
    • only 2 stock prices at the END of that period
Binomial Trees

- Asset Price Movements
  - divide the time from t to T into small intervals $\Delta t$
  - in each time interval, assume the asset’s price $S$ can move UP ↑
    - by a proportional amount $u$
  - move DOWN ↓
    - by a proportional amount $d$

Tree Parameters

- What?
  - $p$, $u$, and $d$
- Parameter values?
  - tree must give correct values
    - for the mean & standard deviation
    - of the stock price changes
    - in a risk-neutral world
- Simplification
  - tree is recombining: $u = 1/d$

Risk-Neutral Valuation

- Assumption
  - no arbitrage opportunity exists
- Basic idea
  - assume a binomial tree for asset price movements
  - create a riskless portfolio
    - stock plus option
  - riskless portfolio always possible with binomial tree
  - value the portfolio
    - if riskless, then risk-neutral valuation is OK
- Reference
  - Cox-Ross-Rubinstein (Journal of Financial Economics)
Risk-Neutral Valuation 3

- Riskless Portfolio
  - Portfolio
    - LONG \( \Delta \) shares
    - LONG 1 put option (why?)
    - \( S = 20 \)
    - \( \$18.18\Delta + 2.82 \)
  - Portfolio is riskless
    - if \( 22\Delta = 18.18\Delta + 2.82 \) i.e \( \Delta = 0.738 \)
    - LONG 0.738 shares and LONG 1 put option

Risk-Neutral Valuation 4

- Value of the entire (riskless) portfolio
  - in 3 months
    - if the stock price moves up:
      - \( \times 22 \times 0.738 + 0 = \$16.24 \)
    - if the stock price moves down:
      - \( \times 18.18 \times 0.738 + 2.82 = \$13.42 + 2.82 = \$16.24 \)
  - today
    - PV of \$16.24 at the risk-free rate (why?)
    - if annual continuously-compounded risk-free rate is 12%, portfolio is worth: \( \$16.24 e^{-0.12 \times 0.25} = \$15.76 \)

Risk-Neutral Valuation 5

- Value of the Option Today
  - Entire portfolio
    - is worth \$15.76
  - Shares
    - are worth \( 0.738 \times 20 = \$14.76 \)
  - Value of the put option
    - is therefore \$15.76 - \$14.76 = \$1.00

Risk-Neutral Valuation 6

- Generalization (H7 Fig. 11.2; H8 Fig. 12.2)
  - derivative
    - value \( f \)
    - expires at time \( T \)
    - is dependent on a stock

Risk-Neutral Valuation 7

- Riskless portfolio
  - LONG \( \Delta \) shares and SHORT 1 derivative
    - \( \Delta S = f \)
    - \( S_u \Delta - f_u \)
    - \( S_d \Delta - f_d \)
    - riskless
      - if \( S_u \Delta - f_u = S_d \Delta - f_d \) or \( \Delta = \frac{f_d - f_u}{S_u - S_d} \)

Risk-Neutral Valuation 8

- Value of the portfolio at time \( T \):
  - \( (\text{up state}) S_u \Delta - f_u = S_d \Delta - f_d \) (down state)
- Value of the portfolio today:
  - \( (S \Delta - f) e^{-rT} \)
  - and also
  - \( S \Delta - f \)
- Hence
  - \( f = S \Delta - (S \Delta - f) e^{-rT} \)
Risk-Neutral Valuation 9

- Thus:
  \[ f = S \Delta - (Su \Delta - f_u) e^{-rT} \]
  \[ \Delta = \frac{f_u - f_d}{Su - Sd} \]
- Substituting for \( \Delta \), we obtain
  \[ f = [p f_u + (1-p) f_d] e^{-rT} \]
  \[ p = \frac{e^{-rT} - d}{u - d} \]

**Irrelevance of Stock’s Expected Return**

- When valuing an option in terms of the underlying stock,
- the expected return on the stock is irrelevant

Original Example Revisited

- Call, H8 Fig. 12.1 (S = 20; X = 21; \( \Delta t = T = 3 \) months)
  \[ Su = 22 \]
  \[ f_u = 1 \]
  \[ Sd = 18 \]
  \[ f_d = 0 \]
- risk-neutral probabilities:
  \[ p = \frac{e^{0.25} - d}{u - d} = \frac{e^{0.12} - 0.9}{1.1 - 0.9} = 0.6523 \]

Original Example Revisited 2

- H8 Fig. 12.1
  \[ Su = 22 \]
  \[ f_u = 1 \]
  \[ Sd = 18 \]
  \[ f_d = 0 \]
- Value of the option
  \[ c = e^{-0.12 \times 0.25} \times [0.6523 \times \$1 + 0.3477 \times \$0] = \$ 0.633 \]

Risk-Neutral Valuation 10

- Interpretation
  \[ f = [p f_u + (1-p) f_d] e^{-rT} \]
  \( p \) and \( (1-p) \) can be interpreted as the risk-neutral probabilities of up & down movements
- Value of a derivative
  \( = \) its expected payoff
  \- in a risk-neutral world
  \- discounted
  \- at the risk-free rate

Original Example Revisited 3

- Key Result
  - risk-neutral valuation ("revisited")
  - coincides with the ("original") no-arbitrage valuation.
- Generalization
  - in general
  - when pricing derivatives
  - using risk-neutral valuation
  - is ok
Original Example Revisited 4

• Valuing the Stock
  • in a risk-neutral world
    – stock must also earn the risk-free rate
  • consequence
    – Since $p$ is a risk-neutral probability
    – $20 \times e^{0.12 \times 0.25} = 22 \times p + 18 \times (1 - p)$
    – $p = 0.6523$

A Two-Step Call Option Example

• $H8$ Fig. 12.3 (X=21; u=1.1; d=0.9; T=6 months)

A Two-Step Call Option Example 2

• Call value (Fig. 12.4; X=21)

A Two-Step Put Option Example

• Fig. 12.7 (X=52; u=1.2, d=0.8, T=2 years)

A Two-Step Put Option Example 2

• European put value (Fig. 12.7; X=52)

A Two-Step Put Option Example 3

• American put value (Fig. 12.8; X=52)
Delta

- **Definition**
  - Delta (Δ) is the ratio
    » of the change in the price of a stock option
    » to the change in the price of the underlying stock

- **Dynamic hedging**
  - The value of Δ varies from node to node
    » Dynamic hedging needed!

Delta 2

- Riskless portfolio at a given node:
  - LONG Δ shares and SHORT 1 derivative
    \[ \Delta S - f_u \]
    \[ \Delta S = f_d \]
  - riskless
    \[ \frac{S_u \Delta - f_u}{S_d \Delta - f_d} = 1 \]

Delta 3

- **European put**
  (S=50; X=52)
  \[ \Delta = -0.4025 \]
  \[ \Delta = 1 \]

- Value of Δ at node B
  \[ = \frac{0-4}{72-48} = -1/6 \] (i.e., short 1 put and short 1/6 share)

- Value at node C
  \[ = \frac{4-20}{48-32} = -1 \] (i.e., short 1 put and short 1 share)

- Value at node A
  \[ = \frac{1.41-9.46}{60-40} = -0.4025 \] (short 1 put and 0.4025 share)