Derivatives

- Lectures 10-12:
  - Part VI: Option Pricing
    » Black-Scholes Formula
    » Binomial Trees
- Last two topics (Weeks 13-14):
  - Part VII: Option pricing & Continuous-time Finance
    » Modeling Asset Prices in Continuous Time
    » Derivation of the Black-Scholes Formula
  - Part VIII: Option pricing – Advanced Topics
    » Hedging; Option Pricing w/ varying risk-free rate

Part VII: Option Pricing & Continuous-Time Finance

Modeling Stock Prices Behavior

- Stochastic Process & Price Modeling
  - continuous time
  - continuous variables
- Markov Processes
  - market efficiency
- Wiener Processes
- Ito Processes & Ito’s Lemma

Stochastic Process & Price Modeling

- Dimensions
  - time
  - discrete measurement intervals vs. continuous flow
  - random variable
  - limited number vs. interval of possible values
- Types of Stochastic Processes
  - Discrete time; discrete variable
  - Discrete time; continuous variable
  - Continuous time; discrete variable
  - Continuous time; continuous variable

Stochastic Process & Price Modeling 2

- Our focus
  - We model stock prices
    with continuous-time, continuous-variable process
  - Why?
    » most useful type of process
    to value derivative securities

From Discrete to Continuous Time

- Normal distribution
  - type of process
    » Discrete time, continuous variable
  - example
    » stock price is currently at $40
    » at the end of 1 year, probability distribution
    » normal distribution $\phi(\mu, \sigma)$, with
      $\phi(\mu, \sigma)$: $\mu$: mean
      $\sigma$: standard deviation
From Discrete to Continuous Time 2

- Taking limits
  - probability distribution of the stock price at the end of
    - 2 years?
    - ½ years?
    - ¼ years?
    - Δ years?
- As Δ → 0,
  - we have defined
    → a continuous variable continuous time process

Markov Processes

- Basic idea
  - a stochastic process "is Markov"
    - if future movements in the random variable
      → depend only on where we are
      → do not depend on the history of how we got there
- Relevance
  - all 4 types of stochastic processes can be Markov
- Our approach
  - assume that the underlying asset’s price
    follows a Markov process

Markov Processes 2

- Weak-Form Market Efficiency
  - definition
    → impossible to produce consistently superior returns
    → with a trading rule
    → based on the past history of stock prices
  - interpretation
    → technical analysis does NOT work
- Observation
  - a Markov process for stock prices
  - is consistent with weak-form market efficiency

Markov Processes 3

- Variances & Standard Deviations
  - changes in successive periods of time
    → independent in Markov processes
  - implication
    → variances are additive
    → standard deviations are NOT additive (why?)
  - example
    → “in 1 year, probability distribution is φ(40,10)”
      - correct to say that variance is 100% per year
      - strictly speaking, NOT correct to say that
        standard deviation is 10% per year

Wiener Processes

- Definition
  - a variable z
    → whose value changes continuously
    → s.t. the change in a small interval of time Δt is Δz
  - follows a Wiener process if
    1. Δz = ε$\sqrt{Δt}$
      where ε is a random drawing from φ(0,1)
    2. the values of Δz
      - for any 2 non-overlapping periods of time
      - are independent

Wiener Processes 2

- Properties of a Wiener Process z(t)
  - moments
    - Mean of [z(T) − z(0)]
      → 0
    - Variance of [z(T) − z(0)]
      → T
    - Standard deviation of [z(T) − z(0)]
      → $\sqrt{T}$
Wiener Processes 3

• Taking Limits
  • what?
    \[ \Delta t \to 0 \implies \Delta x = \sqrt{\Delta t} \]
  • interpretation
    » the corresponding expression involving \( \Delta t \)
    » is true in the limit as \( \Delta t \) tends to zero
    » This is how we define the meaning of \( \Delta x \) & \( \Delta t \)
• stochastic calculus
  » in this respect,
  » stochastic calculus is analogous to ordinary calculus

Generalized Wiener Processes

• Wiener process
  • drift rate ("average change per unit time") = 0
  • variance rate = 1
• Generalized Wiener process
  • idea
    » drift rate & variance rate = any chosen constants
  • definition
    » the variable \( x \) follows a generalized Wiener process
      with drift rate \( a \) & variance rate \( b^2 \)
    » if \( \Delta x = a \Delta t + b \Delta z \) where \( \Delta z \) is Wiener

Generalized Wiener Processes 2

\[ \Delta x = a \Delta t + b \sqrt{\Delta t} \]

• Mean change in \( x \) in time \( T \)
  » \( a \) \( T \)
• Variance of change in \( x \) in time \( T \)
  » \( b^2 T \)
• Std deviation of change in \( x \) in time \( T \)
  » \( b \sqrt{T} \)

Generalized Wiener Processes 3

• Our Example, Revisited
  • data
    » stock price \( S \) starts at $40
    » proba. distribution is \( \phi(40,10) \) at the end of the year
  • if stochastic process for \( S \) is Markov with NO drift
    » then the process is \( dS = 10 \Delta z \)
  • if the stock price is expected to grow by $8
    » on average during the year
    » then the year-end distribution is \( \phi(48,10) \)
    » and the process is \( dS = 8 \Delta t + 10 \Delta z \)

Generalized Wiener Processes 4

• Problem
  • what?
    » generalized Wiener process NOT Appropriate for stocks
  • why?
    » for a stock price, what is constant in a short period
      - is likely to be the expected proportional change
    » for a stock price, uncertainty
      - as to the size of future stock price movements
      - is likely proportional to the stock price level
• Solution
  • Ito processes

Ito Processes

• Characteristic
  • drift rate & variance rate are functions of
    » time
    » & the variable
    » \( dX = a(x,t)dt + b(x,t)dz \)
• Discrete time equivalent
  \[ \Delta x = a(x,t)\Delta t + b(x,t)\varepsilon \sqrt{\Delta t} \]
  • only true in the limit as \( \Delta t \) tends to zero
Ito Processes 2

• Ito Process for Stock Prices
\[ \frac{dS}{S} = \mu dt + \sigma dz \]
• where \( \mu \) is the expected return
• \( \sigma \) is the volatility.
• Discrete time equivalent
\[ \frac{\Delta S}{S} = \mu \Delta t + \sigma \sqrt{\Delta t} \]

Ito’s Lemma

• Key result
  • If we know the stochastic process followed by \( x \),
    • Ito’s lemma tells us
      • the stochastic process followed by a function \( G(x, t) \)
• Why do we care?
  • a derivative security is a function
    » of the price of the underlying
    » & of time
  • thus, Ito’s lemma plays an important part
    » in the analysis of derivatives

Ito’s Lemma 2

Suppose
\[ dx = a(x, t)dt + b(x, t)dz \]

We have
\[ dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} b \right) dt + \frac{\partial G}{\partial x} b \, dz \]
For a general function \( G(x, t) \). This is Ito’s lemma.

Ito’s Lemma 3

• Application to a Stock Price Process

The stock price process is
\[ dS = \mu S \, dt + \sigma S \, dz \]
For a function \( G \) of \( S \) & \( t \)
\[ dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz \]

Ito’s Lemma 4

• Examples
  1. forward price of a stock for a contract maturing at time \( T \)
\[ G = S e^{(r-T)} \]
\[ dG = (\mu - r)G \, dt + \sigma G \, dz \]
  2. \( G = \ln S \)
\[ dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dz \]

Black-Scholes Analysis

• Lognormal distribution
  • expected return
  • variance
• Markov Processes
  • market efficiency
• Wiener Processes
• Ito’s Lemma
The Lognormal Property

• Definition
  • In $S_t$ follows a generalized Wiener process with drift $\mu - \sigma^2/2$

• Implications
  \[ \ln S_t \sim \text{generalized Wiener process with drift} \mu - \sigma^2/2 \]  
  \[ \ln S_t \sim \phi \left( \frac{\ln S_0 - T \mu}{\sqrt{T}}, \frac{\sigma}{\sqrt{T}} \right) \] (13.2) p.278
  \[ \ln S_t \sim \phi \left( \frac{\ln S_0 - T \mu}{\sqrt{T}}, \frac{\sigma}{\sqrt{T}} \right) \] (13.3) p.278

• Put differently
  • $S_T$ is lognormally distributed
  • since the logarithm of $S_T$ is normal

The Lognormal Property 2

• Lognormal Distribution

\[ E(S_T) = S e^{\mu(T-t)} \]
\[ \text{var}(S_T) = S^2 e^{2\mu(T-t)} (e^{\sigma^2(T-t)} - 1) \]

The Lognormal Property 3

• Continuously Compounded Return, $\eta$

\[ S_T = S e^{\eta(T-t)} \]

\[ \eta = \frac{1}{T-t} \ln \frac{S_T}{S} \]

\[ \eta \approx \phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T-t}} \right) \]

The Lognormal Property 4

• The Expected Return
  – Two possible definitions:
  • $\mu$ is the arithmetic average
    • of the returns realized
    • in many short intervals of time
  • $\mu - \sigma^2/2$ is a geometric average
    • is the expected continuously compounded return
    • realized over any finite period of time

The Lognormal Property 5

• The Volatility
  – definition
    • The volatility, $\sigma$ is the standard deviation
    • of the continuously compounded rate of return
    • in 1 year
  – approximation
    • standard deviation of the proportional change
    • in 1 year

The Lognormal Property 6

• Estimating Volatility from Historical Data

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of $\tau$ years
2. Define the continuously compounded return as: $\eta_i = \ln \left( \frac{S_i}{S_{i-1}} \right)$
3. Calculate the standard deviation ($s$) of the $\eta_i$’s
4. Calculate the annualized (historical) volatility: $s^* = \frac{s}{\sqrt{\tau}}$
Concepts Underlying Black-Scholes

- Assume constant risk-free interest rate
- Risk-neutral valuation
  - Option price & stock price depend on the same underlying source of uncertainty, \( dz \)
  - Investor can form a portfolio consisting of the stock & the option which eliminates this source of uncertainty
  - That portfolio is instantaneously riskless & thus must instantaneously earn the risk-free rate
- This leads to the B&S differential equation

Derivation of the Black-Scholes Differential Equation

\[
\Delta S = \mu S \Delta t + \sigma S \Delta z
\]
\[
\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z
\]

We set up a portfolio consisting of
- 1: derivative
  \[ \frac{\partial f}{\partial S} : \text{shares} \]

Derivation of the Black-Scholes Differential Equation 2

The value of the portfolio \( \Pi \) is given by
\[
\Pi = -f + \frac{\partial f}{\partial S} S
\]
The change in its value in time \( \Delta t \) is given by
\[
\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S
\]

Derivation of the Black-Scholes Differential Equation 3

The return on the portfolio must be the risk-free rate.
Hence \( \Delta \Pi = r \Delta t \)
We substitute for \( \Delta f & \Delta S \) in these equations to get the Black-Scholes differential equation:
\[
\frac{\partial f}{\partial t} + r S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

The B&S Differential Equation

- Any security
  - whose price is dependent on the stock price
  - satisfies the differential equation
- The particular security being valued
  - is determined by the boundary conditions of the differential equation
  - e.g. \( \text{Max}(S-K,0) \) for \( t=T \)

Example: Forward Contract

- Boundary condition
  - \( f = S - K \) when \( t = T \)
- Solution to the equation
  \[ f = S - K e^{-r(T-t)} \]
- It is not usually that easy to solve!
  - Risk-Neutral Valuation makes life easier.
**Risk-Neutral Valuation**

- The variable $\mu$ does **NOT** appear in the Black-Scholes equation.
- The B&S equation is independent of investors’ risk preferences.
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world.
- Leads to principle of risk-neutral valuation.

**Applying Risk-Neutral Valuation**

1. Assume that the expected return from the stock price is the risk-free rate.
2. Calculate the expected payoff from the option.
3. Discount at the risk-free rate.

**Example: R-N Value of Forward**

- Payoff from Forward: $S_T - K$
- R-N forward value: $f = e^{-r(T-t)} \hat{E}(S_T - K)$
- R-N Stock: $\hat{E}(S_T) = Se^{(r-\frac{\sigma^2}{2})(T-t)}$
- So, $f = S - Ke^{(r-\frac{\sigma^2}{2})(T-t)}$
- Consistent with Chapter 5 (H7) & Chapter 3 (H6).

**Example: R-N Value of Call Option**

- For a European call option, $c = e^{-r(T-t)} \hat{E} [\max(S_T - X, 0)]$
- In a risk-neutral world, $S_T$ will earn the risk-free rate on average, so $\ln(S_T) = \phi [\ln(S) + (r - \frac{\sigma^2}{2})(T-t), \sigma \sqrt{T-t}]$.
- Solving the expectations integral, you get $\hat{E}[^*]$

**Black-Scholes formula**

- Black-Scholes gives price of European call
  $$c = e^{-r(T-t)} [N(d_1)S - N(d_2)X]$$
  where
  $$d_1 = \frac{\ln(S/X) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$
  $$d_2 = \frac{\ln(S/X) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$
  $N(z)$ is standard normal.
- $N(d_1)$ is probability of exercise.
- $XN(d_2)$ is expected pay-out at exercise.
- $SN(d_2)e^{r(T-t)}$ is expected value of the stock price, if exercised.

**Black-Scholes formula 2**

$$c = e^{-r(T-t)} [SN(d_1)e^{r(T-t)} - XN(d_2)]$$

- $N(z)$ is standard normal.
- $N(d_1)$ is probability of exercise.
- $XN(d_2)$ is expected pay-out at exercise.
- $SN(d_2)e^{r(T-t)}$ is expected value of the stock price, if exercised.
Black-Scholes Properties

- When $S$ gets very large,
  \[ N(d_1) = N(d_2) = 1, \text{ and } c = S - X e^{-r(T-t)} = \text{value of forward} \]

- When $\sigma$ gets very small, again
  \[ N(d_1) = N(d_2) = 1, \text{ and } c = S - X e^{-r(T-t)} \]

European Put Options

- Just use put-call parity. You’ll get,
  \[ p = e^{-(r-t)} [X N(-d_2) - S N(-d_1) e^{r(T-t)}] \]

- The normal distribution is symmetric, so
  \[ N(-d_1) = [1 - N(d_1)] \]

Example 13.6 *(12.7 in H6)*: B-S Pricing

- The stock price today is $42. There is 6 months to expiration. The exercise price is $40. The risk-free rate is 10% and the volatility is 20% per year.
- What is the price of a European call?
- What is the price of a European put?
- Get $N(d_1)$ and $N(d_2)$ from pp. 800-801 *(H7)*.

Warrants & Dilution

- When a regular call option is exercised the stock that is delivered must be purchased in the open market
- When a warrant is exercised new treasury stock is issued by the company
- This will dilute the value of the existing stock
- One valuation approach is to assume that all equity (warrants + stock) follows a geometric Brownian motion, and then use Black-Scholes.

Implied Volatility

- Black-Scholes: $c = BS(S,X,r,T-t,\sigma)$
- Only $\sigma$ is unobserved and must be estimated.
- Call price is observed in market, thus we can back out the so-called B-S implied volatility, theoretically: $\sigma^* = BS^{-1}(S,X,r,T-t,c)$
- No closed-form solution available, but it can be calculated by trial-and-error.

Implied Volatility (cont.)

- There’s a 1-to-1 relationship between the B-S price and IV, and it can be used to estimate the price of one option from another.
- IV’s are used to monitor the market’s view on volatility, as it changes over time.
- Composite IV’s are calculated for a given stock using options with different $X$ and $T-t$. 
Measuring Volatility

- Volatility is much larger on trading days than on non-trading days.
- Black-Scholes is sometimes adjusted using 252 (trading) days per year for volatility and 365 days per year for risk-free interest rate.
- The difference is only significant for short time-to-maturity options.

Dividends

- View dividends as a riskless component of the stock price.
- Use Black-Scholes, but set $S = \text{Today's stock price} - \text{PV of dividends}$
- Example 13.10 (H7): Suppose $S = X = 40$, $r = 9\%$, $\sigma = 30\%$, $\$0.50$ dividends in 2 and 5 months, $T-t = 6$ months. What is the price of a European call?

American Calls

- An American call on a NON-dividend-paying stock should NEVER be exercised early, so $C = c$.
- An American call on a dividend-paying stock should ONLY ever be exercised immediately prior to an ex-dividend date.

Black’s Approach to Dealing with Dividends in American Call Options

- Example 13.10 (H7): Suppose $S = X = 40$, $r = 9\%$, $\sigma = 30\%$, $\$0.50$ dividends in 2 and 5 months, $T-t = 6$ months. What is the price of an American call?
- The exact RGW formula in Appendix 12.A gives a slightly different result. It allows for the investor to decide on exercise at the ex-dividend date. You are not responsible for it.

Options on Stock Indices, Currencies and Futures Contracts

Chapter 14
Recap 1: Black-Scholes

- In Black-Scholes, the underlying asset pays no income during the life of the option

\[ c = e^{-r(T-t)}[SN(d_1) - XN(d_2)] \]

where

\[ d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

Recap 2: Discrete Dividends

- We can easily adjust Black-Scholes to account for discrete dividend payments by subtracting from the stock a risk-free component equal to the PV of the future dividends.

European Options on Stocks Paying Continuous Dividends

We get the same probability distribution for the stock price at time \( T \) in each of the following cases:

1. The stock starts at price \( S \) and provides a continuous dividend yield \( q \).
2. The stock starts at price \( Se^{-q(T-t)} \) and provides no income.

Stock Price Distributions

- Recall the distribution of log-stock prices

\[ \ln(S_T) = N[\ln(S) + (\mu - \sigma^2/2)(T-t), \sigma \sqrt{T-t}] \]

- If continuous dividend yield is paid

\[ \ln(S_T) = N[\ln(S) + (\mu - q - \sigma^2/2)(T-t), \sigma \sqrt{T-t}] \]

- Equivalently, if initial stock price is \( Se^{-q(T-t)} \)

\[ \ln(S_T) = N[\ln(Se^{-q(T-t)}) + (\mu - \sigma^2/2)(T-t), \sigma \sqrt{T-t}] \]

Trick to Extend Black-Scholes

- When pricing derivatives on an underlying asset with continuous dividend yield, \( q \),
  - we can value European options
  - by reducing the stock price to \( Se^{-q(T-t)} \) & then behaving as though there is NO dividend.

European Options on Stocks Paying Continuous Dividends

\[ c = Se^{-r(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2) \]

where

\[ d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\ln(S/X) + (r - q - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]
Bounds for Option Prices, w/ q>0

• European call prices w/ continuous dividends using Chapter 7 formula
  \[ c > S e^{-r(T-t)} - X e^{-r(T-t)} \]
• European put prices w/ continuous dividends
  \[ p > X e^{-r(T-t)} - S e^{-q(T-t)} \]

Call-Put Parity, w/ q > 0

• Again, just substitute \( S e^{-q(T-t)} \) for \( S \) in the Chapter 9 PCP formula to get
  \[ p + S e^{-q(T-t)} = c + X e^{-r(T-t)} \]
• Use call-put parity (PCP) to find put option value
  \[ p = X e^{-r(T-t)} N(-d_2) - S e^{-q(T-t)} N(-d_1) \]

Index Options

• Option contracts are on 100x the index
• The most popular underlying indices are the S&P 100 (OEX), the S&P 500 (SPX), & the Major Market Index (XMI)
• S&P 100 and XMI options are American, S&P 500 options are European.
• Continuous dividend framework useful for valuation of European index options.

Using Index Options for Portfolio Insurance

• Suppose the value of the index is \( S \) & the strike price is \( X \)
• If a portfolio has a \( \beta \) of 1.0, the portfolio manager BUYS 1 put option for each 100S dollars held
• If the \( \beta \) is NOT 1.0, the portfolio manager BUYS \( \beta \) put options for each 100S dollars held
• In both cases, \( X \) is chosen to give the appropriate insurance level

Example: Portfolio Insurance w/ \( \beta=1 \)

• The value of a well diversified stock portfolio is $500,000. The index is at 250 points. Suppose the desired insurance value is $480,000 in three months.
• Insurance: Buy 20 put options with X=240.
• If index falls to 225, the stocks will be worth about $450,000, but the options will be worth 20*(240-225)*100 = $30,000, thus keeping the insurance value intact.
Example: Portfolio Insurance w/ $\beta \neq 1$

• First find relationship between end-of-period index value and end-of-period portfolio value.
• Then pick strike price for index options corresponding to the desired insurance value of the portfolio.

Valuing European Index Options

We can use the formula for an option on a stock paying a continuous dividend yield

• Set $S =$ current index level
• Set $q =$ average dividend yield per year expected during the life of the option
• Be careful with seasonality in dividend payments.

Example 15.1 (H7)

• What is the value of a European call option on the S&P500 with two months to maturity, when the current value of the index is 930, the strike price is 900, the risk-free rate is 8%, volatility is 20%, and dividend yields of 0.2% and 0.3% are expected in the first and second month respectively?

• Answer: $T = 2/12 = 0.167; q =$ dividend yield = 0.5% ($=0.2%+0.3%$); $d_1 = 0.544; d_2 = 0.4328; N(d_1) = 0.7069; N(d_2) = 0.6782; c = 51.83 \Rightarrow$ contract cost $= 5,183$

Currency Options

• Currency options trade on the Philadelphia Exchange (PHLX)
• They are used by corporations to hedge their FX exposure
• The size of a contract depends on the currency
  • Examples: 62,500 GBP, 125,000 CHF
• There also exists an active over-the-counter (OTC) market

Valuing European Currency Options

• A foreign currency is an instrument that provides a continuous income equal to the foreign risk-free rate ($r_f$)
• We can use the formula for an option on a stock paying a continuous dividend yield:
  • Set $S =$ current exchange rate
  • Set $q =$ $r_f$
  • Note that the income earned in the domestic currency is $r_fS$ showing that $r_f$ is analogous to $q$

European Currency Options

• Pricing formulas
  \[ c = S e^{-r_f(T-t)} N(d_1) - X e^{-r_f(T-t)} N(d_2) \]

  where

  \[ d_1 = \frac{\ln(S/X) + (r_r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

  \[ d_2 = \frac{\ln(S/X) + (r_r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]
Put-Call Parity for Currency Options

- Again, just substitute for $S$ in Chapter 7 formula to get new parity
  \[ S e^{-r(T-t)} + p = c + X e^{-r(T-t)} \]
- Use put-call parity to find put price
  \[ p = X e^{-r(T-t)} N(-d_2) - S e^{-r(T-t)} N(-d_1) \]

Simplification using Forward Price

- Recall we have \( F = S e^{r(T-t)} \)
- Thus we can simplify Black-Scholes to write
  \[ c = e^{-r(T-t)} [FN(d_1) - FN(d_2)] \]
  where
  \[ d_1 = \frac{\ln(F/X) + (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]
  \[ d_2 = \frac{\ln(F/X) - (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

Futures Options (Chapter 16)

- When a CALL is exercised, the holder acquires a LONG position in the futures plus a cash amount equal to the excess of
  - the futures price over the strike price
- When a PUT is exercised the holder acquires a SHORT position in the futures plus a cash amount equal to the excess of
  - the strike price over the futures price

Example

- An investor has a September futures call option on 25,000 pounds of copper with a strike price of 70 cents per pound. The current futures price of September copper is 80 cents. If exercised, the investor receives $2,500 plus a long futures position in 25,000 pound of September copper. The futures position can be closed out immediately at no cost.

Futures Options [cont.]

- Underlying is a futures contract which expires a few days after the expiration of the option.
- Very popular due to
  - Cheaper delivery (hogs vs. hog futures)
  - Often settled in cash (futures position closed out)
  - Traded side-by-side with underlying

Valuing European Futures Options

- We can use the formula for an option on a stock which pays a continuous dividend yield:
  - Set \( S = \) current futures price \( (F) \)
  - Set \( q = \) domestic risk-free rate \( (r) \)
- Setting \( q = r \) ensures that the expected growth of \( F \) in a risk-neutral world is \( \dot{0} \)
Black’s Formula

- The formulas for European options on futures are sometimes referred to as Black’s formulas.

\[ c = e^{-\sigma^2 \frac{T}{2} \left( N(d_1) - X N(d_2) \right)} \]
\[ p = e^{-\sigma^2 \frac{T}{2} \left( \ln \left( \frac{F}{X} \right) + \sigma^2 \frac{T}{2} \right)} \]

where
\[ d_1 = \frac{\ln \left( \frac{F}{X} \right) + \sigma \sqrt{T-t}}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\ln \left( \frac{F}{X} \right) - \sigma \sqrt{T-t}}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

Growth Rates For Futures Prices

- A futures contract requires NO initial investment.
- In a risk-neutral world the expected return should be 0.
- The expected growth rate of the futures price is therefore 0.
- The futures price can therefore be treated like a stock paying a dividend yield of r.

Summary of Put-Call Parities

Indices:
\[ c + X e^{-r(T-t)} = p + S e^{r(T-t)} \]

Foreign exchange:
\[ c + X e^{-r(T-t)} = p + S e^{r(T-t)} \]

Futures:
\[ c + X e^{-r(T-t)} = p + F e^{r(T-t)} \] (12.13) p.278

Summary of other Key Results

- We can treat stock indices, currencies, & futures like a stock paying a continuous dividend yield of q.
  - For stock indices, \( q = \text{average dividend yield per year on the index over the option life} \)
  - For currencies, \( q = r_f \)
  - For futures, \( q = r \)