Duration

- Weighted average time to maturity using the relative present values of cash flows as weights
  - takes into account the effects of differences in both coupon rates and differences in maturity
- Based on elasticity of bond price with respect to interest rate: derivative scaled by bond price
  - simple differentiation of price function wrt yield
- The units of duration are years: cash-flow adjusted maturity
Macaulay Duration

- Formula

\[
D = \frac{\sum_{t=1}^{N} CF_t \times DF_t \times t}{\sum_{t=1}^{N} CF_t \times DF_t} = \frac{\sum_{t=1}^{N} PV_t \times t}{\sum_{t=1}^{N} PV_t}
\]

Where

\(D\) = Macaulay duration (in years)
\(t\) = number of periods in the future
\(CF_t\) = cash flow to be delivered in t periods
\(N\) = time-to-maturity
\(DF_t\) = discount factor

Duration

- Since the price (P) of the bond equals the sum of the present values of all its cash flows, we can state the duration formula another way:

\[
D = \frac{\sum_{t=1}^{N} PV_t \times t}{P}
\]

- Notice the weights correspond to the relative present values of the cash flows
Semiannual Cash Flows

• Important reminders:
  – we must express \( t \) in years (for consistency)
  – the present values are computed using the appropriate \textit{periodic} interest rate.

• For semiannual cash flows, Macaulay duration, \( D \) is equal to:
\[
D = \frac{\sum_{i=t/2}^{N} \frac{CF_i \times t}{(1 + R/2)^{2t}}}{\sum_{i=t/2}^{N} \frac{CF_i}{(1 + R/2)^{2t}}}
\]

Duration of Zero = Its Maturity

• For a zero-coupon bond, Macaulay duration equals maturity
  – 100\% of its present value is generated by the payment of the face value, at maturity

• For all other bonds, duration < maturity
  – can you see why?
Example

• Consider a 2-year, 8% coupon bond face
  – value of $1,000 and yield-to-maturity of 12%
  – coupons are paid semi-annually: how many periods?

• Therefore, each coupon payment is $40 and the
  per period YTM is \((1/2) \times 12\% = 6\%\)
  – yield already adjusted for semi-annual compounding

• Present value of each cash flow equals \(CF_t \div \) \((1 + 0.06)^t\) where \(t\) is the period number

<table>
<thead>
<tr>
<th>(t) years</th>
<th>CF&lt;sub&gt;t&lt;/sub&gt;</th>
<th>PV(CF&lt;sub&gt;t&lt;/sub&gt;)</th>
<th>Weight (W)</th>
<th>W \times years</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>40</td>
<td>37.736</td>
<td>0.041</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>40</td>
<td>35.600</td>
<td>0.038</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>40</td>
<td>33.585</td>
<td>0.036</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>1,040</td>
<td>823.777</td>
<td>0.885</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>P = 930.698</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Duration: Special Cases

• Consols (“perps” in the jargon): perpetual bonds issued by governments, e.g., UK
  – if c and y are the coupon and yield: its price?
  – Maturity of a consol: \( M = \infty \).
  – Duration of a consol: \( D = 1 + 1/R \)

• Floating rate note or any other variable-interest fixed-income instrument
  – duration = length of reset period
  – inverse floater: tricky and surprising...

Properties of Duration

• Duration and maturity
  – D increases with M, but at a decreasing rate

• Duration and yield-to-maturity
  – D decreases as yield increases

• Duration and coupon interest
  – D decreases as coupon increases

• Duration is additive: how fortunate!
Economic Interpretation

• Duration is a measure of interest rate sensitivity or elasticity of a liability or asset:

\[
\frac{\Delta P}{P} \div \frac{\Delta R}{(1+R)} = -D
\]

• Or equivalently,

\[
\frac{\Delta P}{P} = -D\frac{\Delta R}{(1+R)} = -MD \times \Delta R
\]

where MD is modified duration

Duration and Yield Sensitivity

\[
\frac{dP}{dy} = -P \cdot \text{Modified Duration}
\]

Duration is related to the rate of change in a bond's price as its yield changes: linear (first-order) approximation of price changes in yield - let's see what the problem is…
Predicting Price Changes

1. Find Macaulay duration of bond
2. Find modified duration of bond
3. When interest rates change, the change in a bond’s price is related to the change in yield according to

\[ \frac{\Delta P}{P} \times 100 = -D_m^* \times \Delta y \times 100 \]

- Find percentage price change of bond
- Find predicted dollar price change in bond
- Add predicted dollar price change to original price of bond
  \[ \Rightarrow \] Predicted new price of bond (…or use XLS)

Predicting Price Changes

• To estimate the change in price, we can express the previous equation as:

\[ \Delta P = -D[\Delta R/(1+R)]P = -(MD) \times (\Delta R) \times (P) \]

• Note the direct linear relationship between \( \Delta P \) and -D: what does this imply?
Dollar Duration

- Dollar duration equals modified duration times price
  - Dollar duration = MD × Price
- Using dollar duration, we can compute the change in price as
  \[ \Delta P = -\text{Dollar duration} \times \Delta R \]
- Allows us to compute the direct price impact of yield or interest-rate changes

Semi-annual Coupon Payments

- With semi-annual coupon payments the percentage change in price is given by
  \[ \frac{\Delta P}{P} = -D\left[\frac{\Delta R}{1+(R/2)}\right] \]
- As always, for more frequent compounding we need to adjust
  - yield
  - number of (time) periods
Managing Interest-Rate Risk

- Measuring interest-rate exposure
  - duration:
  - potential problems?
- Managing interest-rate exposure:
  - principle:
  - implementation?
- Overall objective:

Immunization

- Matching the *maturity* of an asset investment with a future payout responsibility
  - does not necessarily eliminate interest rate risk
  - but introduces fundamental principle: match assets and liabilities along risk sensitivity
- Matching *durations* will immunize against changes in interest rates
  - why?
Small Business Lending: in Ks

- Consider three loan plans, all of which have maturities of 2 years.
  - amount is $1,000 and the current interest rate is 3%.
- Loan #1 is a two-payment loan with two equal payments of $522.61 each.
- Loan #2 is structured as a 3% annual coupon bond.
- Loan #3 is a discount loan, which has a single payment of $1,060.90.

Duration as Index of Interest Rate Risk

<table>
<thead>
<tr>
<th>Loan Value</th>
<th>Yield</th>
<th>2%</th>
<th>3%</th>
<th>ΔP</th>
<th>N</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal Payment</td>
<td>$1014.68</td>
<td>$1000</td>
<td>$14.68</td>
<td>2</td>
<td>1.493</td>
<td></td>
</tr>
<tr>
<td>3% Coupon</td>
<td>$1019.42</td>
<td>$1000</td>
<td>$19.42</td>
<td>2</td>
<td>1.971</td>
<td></td>
</tr>
<tr>
<td>Discount</td>
<td>$1019.70</td>
<td>$1000</td>
<td>$19.70</td>
<td>2</td>
<td>2.000</td>
<td></td>
</tr>
</tbody>
</table>
Balance-Sheet Immunization

• Duration is a measure of the interest rate risk exposure for an FI
• If the durations of liabilities and assets are not matched, then there is a risk that
  – adverse changes in the interest rate will increase the present value of the liabilities
  – more than the present value of assets is increased
• Result?

Duration Gap

• A simple FI balance sheet:
  – a 2-year coupon bond is the only loan asset (A)
  – a 2-year certificate of deposit is the only liability (L)
• If the duration of the bond is 1.8 years, then:
  Maturity gap: $M_A - M_L = 2 - 2 = 0$, but
  Duration Gap: $D_A - D_L = 1.8 - 2.0 = -0.2$
  – Deposit has greater interest rate sensitivity than the bond, so DGAP is negative
  – FI exposed to rising interest rates
Immunizing the Balance Sheet

• Duration Gap: the DGAP
  – From the balance sheet, \( E = A - L \); therefore,
  \[ \Delta E = \Delta A - \Delta L \]
  – reflects what management priority?
• Like for changes in bond prices, we can find the change in value of equity using duration.
  \[ \Delta E = \left[ -D_A A + D_L L \right] \Delta R/(1+R), \text{ or} \]
  \[ \Delta E = -[D_A - D_L k] A(\Delta R/(1+R)) \]

Duration and Immunizing

• The formula shows 3 effects:
  – Leverage adjusted D-Gap
  – The size of the FI
  – The size of the interest rate shock
• How does each factor affect the value of a financial institution?
  – leverage:
  – size:
  – interest-rate shock:
An Example

• Suppose $D_A = 5$ years, $D_L = 3$ years and rates are expected to rise from 10% to 11%.
  – Rates change by 1%
  – balance sheet: $A = 100$, $L = 90$ and $E = 10$.
• Find change in $E$
  $$\Delta E = -[D_A - D_L k] A \Delta R / (1+R)$$
  $$= -[5 - 3(90/100)]100[.01/1.1] = -$2.09.$$
• Methods of immunizing balance sheet
  – simply adjust $D_A$, $D_L$ or $k$: what is least expensive?

Immunization and Regulation

• Regulators set target ratios for an FI’s capital (net worth): immunize capital ratio
  – Capital (Net worth) ratio = $E/A$
• If target is to set $\Delta (E/A) = 0$: $D_A = D_L$
  – accept without proof
• But, to set $\Delta E = 0$:
  – $D_A = kD_L$
• Who should managers make happy?
  – shows what?
Limitations of Duration

• Immunizing the entire balance sheet need not be costly: why not that expensive for FIs?
• Duration can be employed in combination with hedge positions to immunize
• Immunization is a dynamic process since duration depends on instantaneous R
• Large interest rate changes not accurately captured
  – Convexity
• More complex if nonparallel shift in yield curve
  – stylized facts on term structure of interest rates (TSIR)?

Convexity

• The duration measure is a linear approximation of a non-linear function.
  – If there are large changes in R, the approximation is much less accurate.
  – All fixed-income securities are convex.
• Convexity is desirable, but
  – greater convexity causes larger errors in the duration-based estimate of price changes.
Convexity

- Remember calculus? linear approximations
  - notice that duration involves only the first derivative of the price function.
- Improve on the estimate using a second-order Taylor expansion: higher-order approximation
  - in practice, the expansion rarely goes beyond second order (using the second derivative).
- This second order expansion is the convexity adjustment.

\[
\frac{\Delta P}{P} = -D_M \Delta y
\]
Modified Duration & Convexity

• Second-order approximation using convexity:
  \[ \frac{\Delta P}{P} = -D[\Delta R/(1+R)] + (1/2) \cdot CX \cdot (\Delta R)^2 \]
  or
  \[ \frac{\Delta P}{P} = -MD \cdot \Delta R + (1/2) \cdot CX \cdot (\Delta R)^2 \]
  where MD is modified duration and CX is a measure of the curvature effect.
  – CX = Scaling factor × [capital loss from 1bp rise in yield + capital gain from 1bp fall in yield]

• Commonly used scaling factor is \(10^8\)

Calculation of CX

• Convexity of a 8% coupon, 8% yield, six-year maturity Eurobond priced at $1,000 (why??)
  \[ CX = 10^8[\frac{\Delta P^-}{P} + \frac{\Delta P^+}{P}] \]
  \[ = 10^8[(999.53785-1,000)/1,000 + (1,000.46243-1,000)/1,000)] \]
  \[ = 28 \]
Contingent Claims

• Interest rate changes also affect value of off-balance sheet claims
• Duration gap hedging strategy must include the effects on off-balance sheet items such as
  – futures, options, swaps, caps, and other contingent claims
• In fact: derivatives widely used to adjust duration, i.e., interest-rate exposure

Summary

• Duration and convexity are market-value DCF based measures of interest exposure
  – basis for FIs’ risk and balance-sheet management
• Adjusting duration gap:
  – derivatives
  – product pricing: how to shorten duration?
• Tension between shareholders’ and regulators’ objective: apparent conflict of interest
  – but do shareholders really want to be immunized?