From Yields to Their Analysis

- Review: discrete and continuous spot rates
- Yield and bond price volatility
  - duration
  - convexity
- Introduction to the yield curve
  - revisit yields and their calculation
  - the term structure of interest rates
- Yield curve strategies
Semi-annual Compounding

- Semi-annual compounding: twice a year the bond holder receives a coupon payment
- To obtain the semi-annual compounding rate we have:
  \[ r_2(t,T) = 2 \times \left( \frac{1}{Z(t,T)^{2x(T-t)}} - 1 \right) \]
- where \( Z(t,T) \) is a discount factor

More Frequent Compounding

- Let the discount factor \( Z(t,T) \) be given, and let \( r_n(t,T) \) denote the annualized \( n \)-times compounded interest rate. Then \( r_n(t,T) \) is defined by the equation
  \[ r_n(t,T) = n \times \left( \frac{1}{Z(t,T)^{n(T-t)}} - 1 \right) \]
- Rearranging for \( Z(t,T) \) we obtain
  \[ Z(t,T) = \left( \frac{1 + r_n(t,T)}{n} \right)^{n(T-t)} \]
Continuous Compounding

- The continuously compounded interest rate is obtained by increasing the compounding frequency \( n \) to infinity.
- The continuously compounded interest rate \( r(t, T) \) obtained from \( r_n(t, T) \) for \( n \) that increases to infinity, is given by the formula:
  \[
  r(t, T) = \ln(Z(t, T)) / (T - t)
  \]
  Solving for \( Z(t, T) \) we obtain:
  \[
  Z(t, T) = e^{-r(t, T)(T - t)}
  \]
  where “\( \ln(.) := \log(.) \)” denotes the natural logarithm.
- Remember the properties of log’s?

Discount Factors and Interest Rates

- Note that independently of the compounding frequency, discount factors are the same – otherwise, we would have what?
- Thus some useful identities are:
  \[
  r(t, T) = n \times \ln \left(1 + \frac{r_n(t, T)}{n}\right)
  \]
  \[
  r_n(t, T) = n \times \left( e^{r(t, T)/n} - 1 \right)
  \]
Bond Yield to Maturity

- The price of S1 of an option-free bond as a function of the bond’s yield to maturity \( y \) is

\[
P = \sum_{i=1}^{n} \frac{C_i / 2}{(1 + y)^{i-1+q}} + \frac{1}{(1 + y)^{n-1+q}}
\]

where \( q = \frac{\text{days from settlement to next coupon date}}{\text{days in current coupon period}} \)

Differences in YTM

- Different bonds have different yields
  - Term differences
  - Risk differentials (quality spread)
  - Sector spreads (e.g. industrials v. utilities)
  - Optionality
  - Tax differences
- Benchmark: Treasuries
  - Risky bonds priced at a spread to Treasuries
Yield Curve

• A graph of bond yields to maturity by time to maturity is called a yield curve.

Bond Price Volatility

• Consider only IR as a risk factor
• Longer TTM means higher volatility
• Lower coupons means higher volatility
• Floaters have a very low price volatility
• Price is also affected by coupon payments
• Price value of a Basis Point (PVBP)= price change resulting from a change of 0.01% in the yield.
Price-Yield for Option-Free Bonds

Zero-coupon Example

\[ P_0 = \frac{100}{(1 + y)^T} \quad P_1 = \frac{100}{(1 + y + \Delta y)^T} \]

\[ \frac{P_1 - P_0}{P_0} = \left( \frac{100}{(1 + y + \Delta y)^T} - \frac{100}{(1 + y)^T} \right) \frac{(1 + y)^T}{100} \]
Lessons?

$y=10\%, \Delta y=0.5\%$

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<th>$P_1$</th>
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Approximating Price Changes

We need to find some approximation of price changes in yield:

- How would we go about this task?
- How about a linear approximation?
Focus: Parallel Yield-Curve Shifts

- Current TSoIR

Maturity and Bond-Price Sensitivity

Consider two otherwise identical bonds:

- The long-maturity bond will have much more volatility with respect to changes in the yield
- Larger change wrt discount factor at new rate!

\[ Z(t,T) = \frac{1}{\left(1 + \frac{r_n(t,T)}{n}\right)^{n(T-t)}} \]
Duration

- F. Macaulay (1938)
  - Better measurement than time to maturity.
  - Weighted average of all coupons with the corresponding time to payment.
- YTM bond price \( P(y) = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \)
- Macaulay Duration \( D_{MC} = \frac{1}{P} \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} t = \frac{1}{P} \sum_{t=1}^{T} w_t t \)

suggested weight of each cash-flow date:
- \( w_t = \frac{C_t}{(1+y)^t} / \text{Bond Price: what is the sum of all } w_t \)?

From Price to “Duration”

\[
P = \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \ldots + \frac{C}{(1+y)^n} + \frac{M}{(1+y)^n}
\]

Macaulay Duration =

\[
\frac{1}{P} \left( \frac{1C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{nC}{(1+y)^n} + \frac{nM}{(1+y)^n} \right)
\]
Price Sensitivity: Taking Derivatives

\[ P = \frac{C}{(1 + y)} + \frac{C}{(1 + y)^2} + \ldots + \frac{C}{(1 + y)^n} + \frac{M}{(1 + y)^n} \]

\[ \frac{dP}{dy} = \frac{-C}{(1 + y)^2} + \frac{-2C}{(1 + y)^3} + \ldots + \frac{-nC}{(1 + y)^{n+1}} + \frac{-nM}{(1 + y)^{n+1}} \]

\[ \frac{dP}{dy} = -1 \left( \frac{C}{(1 + y)} + \frac{2C}{(1 + y)^2} + \ldots + \frac{nC}{(1 + y)^n} + \frac{nM}{(1 + y)^n} \right) \]

Duration and IR Exposure

\[ \frac{\Delta P}{P} = -D_{MC} \left[ \frac{\Delta(1 + y)}{1 + y} \right] \]

- The bond price volatility is proportional to the bond’s duration.
- Thus duration is a natural measure of interest rate risk exposure.
Macaulay Duration: Interpretation

\[ D = \sum_{t=1}^{T} t w_t = \frac{1}{\text{Price}} \sum_{t=1}^{T} t \frac{CF_t}{(1 + y)^t} \]

- Duration is the cash-flow weighted sum of times to maturities of each coupon and (final) cash flow.

- What is the duration of a zero coupon bond?

Duration and IR Sensitivity
Modified Duration

\[ D_M = \frac{D_{MC}}{1 + y} \]

\[ \frac{\Delta P}{P} = -D_M \Delta y \]

- The percentage change in bond price is the product of modified duration and the change in the bond’s yield to maturity.

Duration and Yield Sensitivity

\[ \frac{dP}{dy} = - P \cdot \text{Modified Duration} \]

Duration is related to the rate of change in a bond's price as its yield changes: linear (first-order) approximation of price changes in yield - let’s see what the problem is…
Predicting Price Changes

1. Find Macaulay duration of bond
2. Find modified duration of bond
3. When interest rates change, the change in a bond’s price is related to the change in yield according to
   \[ \frac{\Delta P}{P} \times 100 = -D_m^* \times \Delta y \times 100 \]
   – Find percentage price change of bond
   – Find predicted dollar price change in bond
   – Add predicted dollar price change to original price of bond
     ⇒ Predicted new price of bond (…or use XLS)

A Tale of Two Bonds

- Coupon bond with duration 1.8853
- Price (at 5% for 6M = 10.25% p.a.) is $964.5405
- If IR increase by 1bp to 5.01% (= in p.a.-terms?),
  – its price will fall to $964.1942,
  – or 0.359% decline.
- Zero-coupon bond with equal duration must have 1.8853 years to maturity: why?
- At 5% semiannual its price is \($1,000/1.05^{3.7706}$$\approx$831.9623
- If IR increase to 5.01%, the price becomes:
  – \($1,000/1.0501^{3.7706}$$\approx$831.66
  – 0.359% decline.
Duration Determinants

- Duration of a zero-coupon bond equals maturity.
- Duration of a perpetuity is \((1+y)/y\).
- Holding time-to-maturity constant, duration is higher when coupons are lower.
- Holding coupon rate constant duration does not always increase in time-to-maturity!
- Holding other factors constant, duration is higher when YTM is lower.
Example

- A bond with 30-yr to maturity
- Coupon 8%; paid semiannually
- YTM = 9%
- \( P_0 = $897.26 \)
- \( D = 11.37 \) Yrs
- if YTM = 9.1%, what will be the price?
  - \( \Delta P/P = - \Delta y D^* \)
  - \( \Delta P = -(\Delta y D^*)P = -$9.36 \)
  - \( P = $897.26 - $9.36 = $887.90 \)

Usefulness of Duration

- Simple summary statistic of effective average maturity
- Measures sensitivity of bond price to interest rate changes
  - Measure of bond price volatility
  - Measure of interest-rate risk
- Risk management by issues, banks, borrowers:
  - match the duration of assets and liabilities
  - hedge the interest rate sensitivity of an investment
Duration of a Portfolio

- Duration of a portfolio is a weighted average of the duration of assets,
  - weights correspond to the percentage of the portfolio invested in a given security
- The duration of a portfolio of $n$ securities is
  \[ D_w = \sum_{i=1}^{n} w_i D_i \]
  where $w_i$ is the fraction of the portfolio in security $i$, and $D_i$ is the duration of security $i$

Cash Flow Matching and Immunization

- An institution may consider two alternatives to hedge interest rate risk:
  - Cash Flow Matching: buy a set of securities that payoff the exact required cash flow over the period
  - Immunization: portfolio of securities with the same present value and duration of the cash flow needed
- Immunization is preferred to other strategies
  - allows an institution to choose bonds in terms of liquidity and transaction costs benefits
  - generates the desired stream of cash flows
Benefits of Immunization

- Think of two extremes and the effect of a movement in interest rates:
  - 100% in long term bonds: loose money when interest rates go up as bond prices decline
  - 100% investment in cash: loose money when interest rates go down

- Best strategy is in the middle: through duration
  - immunization chooses the adequate mix of the two strategies to hedge against changes in interest rates

Immunization

- Two kinds of Interest rate risk
  - Price risk when interest rate rises
  - Reinvestment risk when interest rate falls

- Immunization offsets these two risks
  - If the duration is set equal to the investment horizon, the two effects exactly cancel out
  - Matching the durations of asset and liability eliminates interest rate risk.
  - Valid for small changes in interest rates
  - Requires rebalancing regularly or when necessary

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Value as of Period 4

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Example: Immunization

- Suppose an insurance company has the following
  - Liability: Payment of $19,487 in seven years
  - Assets: The fund manager decides to hold (1) 3-year zero-coupon bond, and (2) Perpetuity (Assume market interest rate = 10%)

- Solve for the portfolio weights for immunization
  - Duration of liability = 7 years
  - Duration of a portfolio = weighted average of duration of assets in the portfolio = \( w \times 3 + (1 - w) \times 11 \)
    (duration of the perpetuity = \((1+0.1)/0.1 = 11\) years)
  - Match the portfolio duration to 7 years:
    \( w \times 3 + (1 - w) \times 11 = 7 \)
    \( \therefore w = 0.5 \) (3-yr zero), and \((1-w)=0.5 \) (perpetuity)

- After one year, do you have to rebalance it to maintain the immunization strategy? (assume interest rate is still 10%) What should be the new weights for each asset?

Summary

- An introduction to duration analysis
  - Interest rate risk: changes in bond prices arising from fluctuating interest rates (varying YTMs).
  - Ceteris paribus, the longer the time to maturity, the greater the interest rate risk. Ceteris paribus, the lower the coupon rate, the greater the interest rate risk.

- Repayment Risk: debtor might repay the amount owed before maturity

- Default risk: covenants and sinking funds.
  - rating agencies: downgrades and upgrades
  - fixed-income analysis: extracting default probabilities and recovery rates from stock and bond prices
Appendix

• The mathematics of duration analysis
  – derivatives
  – approximating functions
  – Taylor series and approximation
• Correcting for pricing errors
  – second order approximations
  – convexity

Derivatives

• Difference quotients: taking the limit of

\[ m = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \]

one obtains the derivative of \( f(x) \): \( f'(x) \)

• Computing \( f'(x) \)
  is called differentiation
Derivatives

Properties of Derivatives

\[(f(x) + g(x))' = f'(x) + g'(x)\]
\[(f(x) \times g(x))' = f'(x) \times g(x) + f(x) \times g'(x)\]
\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}
\]
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)
\]
Bond Price Change

Modified duration \[ D^* = \frac{D_{MC}}{1+y} \]

Bond-price changes as first-order or higher order approximations? Pro and con?

\[ \Delta P = -D^* \times P \times \Delta y + \frac{1}{2} C \times P \times \Delta y^2 + \ldots \]

Taylor Expansion

• To measure the price response to a small change in risk factor we use the Taylor expansion.
• Initial value \( y_0 \), new value \( y_1 \), change \( \Delta y \):
  \[ y_1 = y_0 + \Delta y \]

\[ f(y_1) = f(y_0) + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)\Delta y^2 + \ldots \]
Pricing Error and Convexity

\[ \frac{\Delta P}{P} = -D_M \Delta y \]

The Second Order: Convexity

\[ C = \frac{d^2 P}{dy^2} \]

Correcting for curvature: second-order approximation of the price-yield relation to improve precision of price-change prediction:
- addresses one problem with duration
- which one is left?

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Convexity

\[ Curv = \frac{d^2 P}{dy^2} \]

\[ C = \frac{d^2 P}{dy^2} \frac{1}{P} = \frac{1}{P_0 (1 + y/2)^2} \left[ \sum_{t=1}^{2T} \frac{t}{2} \frac{t + 1}{2} \frac{C_t/2}{(1 + y/2)^{2t}} \right] \]

Convexity Pitfalls

- A common pitfall is to consider convexity as the change in duration, it is not.
- Many think that since the duration of a zero coupon bond is a constant, regardless of the interest rate, its convexity is zero.
  - this can be somewhat true for dollar convexity which is related to the change in dollar duration.
  - the dollar convexity of a zero coupon is not constant, since it includes the price of the bond.
Measuring Price Changes

- The mathematical principle: Taylor expansion of price-yield function $P$
  $$dP = \frac{dP}{dy} dy + \frac{1}{2} \frac{d^2 P}{dy^2} (dy)^2 + \text{error}$$

- Operational implementation: (modified) duration $D$ and convexity $Conv$
  $$\frac{dP}{P} = -D dy + \frac{Conv}{2} (dy)^2 + \frac{\text{error}}{P}$$

Example

- 10 year zero coupon bond with a semiannual
  $$P = \frac{100}{(1 + 0.03)^{20}} = 55.368$$

- The duration is 10 years, the modified duration is:
  $$D^* = \frac{10}{(1 + 0.03)} = 9.71$$

- The convexity is
  $$C = \frac{1}{P} \frac{d^2}{dy^2} \left( \frac{100}{(1 + 0.5y)^{20}} \right) = 98.97$$
Example

• If the yield changes to 7% the price change

\[ \Delta P = \]

\[ = -9.71 \times 55.37 \times 0.01 + 98.97 \times 55.37 \times 0.01^2 \times 0.5 = \]

\[ = -5.375 + 0.274 = -5.101 \]

\[ \Delta P = \frac{100}{(1 + 0.035)^{20}} - \frac{100}{(1 + 0.03)^{20}} = -5.111 \]

Properties of Convexity

• There are two general rules on convexity:
  – The higher the yield to maturity, the lower the convexity, all else being equal
  – The lower the coupon, the greater the convexity, all else being equal

• To minimize the adverse effects of interest rate volatility for a given portfolio duration
  – seek higher convexity while meeting the other constraints in bond portfolios