RESEARCH STATEMENT

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My principal research interests lie in the representation theory of p-adic groups and the theory of automorphic forms. These areas of mathematics are natural extensions and generalizations of the classical theory of modular forms on the upper half-plane. Automorphic and p-adic representations are of interest not only in their own right, but also because they encode deep number theoretic information. Indeed, the Langlands principle of functoriality conjectures, roughly speaking, that they should correspond to representations of absolute Galois groups of global and p-adic fields, respectively. Moreover, the principle predicts that the irreducible representations of certain pairs of groups should be in correspondence. An example of the power of this theory is Langlands's application of one such correspondence of automorphic representations (the base change lifting) to prove certain cases of Artin's well-known conjecture on the analyticity of L-functions. One focus of my research is the investigation of certain instances of the principle of functoriality for representations of p-adic groups. More specifically, I have been studying the above correspondence in terms of the restrictions of these representations to compact open subgroups.

I am also engaged in research concerning newforms. The classical theory of newforms on the upper half-plane due to Atkin and Lehner has an interpretation in terms of the representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$, and I am currently working to extend the main features of this theory to certain other groups. These results are expected to have applications to the study of automorphic representations of these groups. Also on the automorphic side, I have studied some questions involving Gross's theory of algebraic modular forms for various groups and the Galois representations that are conjectured to correspond to them. In addition, I have worked on several problems concerning local Hecke algebras which arose from my work on algebraic modular forms.

In Section 1, I describe my work concerning local instances of functoriality for p-adic groups, in particular the base change lifting from the unitary group $U(2,1)$ to GL_3 [1] and certain correspondences between the representations of various general linear groups over division algebras [28]. In Section 2, I summarize the questions I have been investigating concerning newforms. My research on algebraic modular forms and Galois representations, as well as the related work on local Hecke algebras, is described in Section 3.

1. Base Change and Other Instances of Functoriality

One goal of my research is to give an explicit description of the local base change lifting of representations of certain p-adic groups. In particular, I am investigating the compatibility of this lifting with the description of representations in terms of their minimal K-types due to Moy-Prasad [31].

Suppose that $\underline{G}$ is a connected reductive group defined over a p-adic field F with residue field k_F. Let E/F be a cyclic extension of degree l, and denote by ε a generator of $\mathrm{Gal}(E/F)$. Let $\underline{\widetilde{G}}$ denote the group $\mathrm{Res}_{E/F}(\underline{G})$ obtained from $\underline{G}$ by restriction of scalars. Let ${}^L\underline{G}$ and ${}^L\underline{\widetilde{G}}$ be the respective L-groups of $\underline{G}$ and $\underline{\widetilde{G}}$. Then the connected component ${}^L\underline{\widetilde{G}}^0$ is isomorphic to a product of l copies of the group ${}^L\underline{G}^0$. The embedding ${}^L\underline{G}^0 \to {}^L\underline{\widetilde{G}}^0$ given by $g \mapsto (g, \ldots, g)$ extends naturally to a map ${}^L\underline{G} \to {}^L\underline{\widetilde{G}}$. The standard base change lifting from $\underline{G}$ to $\underline{\widetilde{G}}$ is the corresponding map of L-packets conjectured to exist by the Langlands principle of functoriality. Essentially, it should take L-packets of representations of $G = \underline{G}(F)$ to those of $\widetilde{G} = \underline{\widetilde{G}}(F) = \underline{G}(E)$ which are ε-invariant.

This map has been shown to exist via global methods in certain instances, e.g., for $\underline{G} = \mathrm{GL}_2$ by Shintani [36] and Langlands [22], for $\underline{G} = \mathrm{GL}_3$ by Flicker [11], for $\underline{G} = \mathrm{GL}_n$ generally by Arthur-Clozel [2], and for unitary groups in two and three variables by Rogawski [35]. All of the above work uses an alternate characterization of the base change lifting via a formula (originally due to Shintani) which relates the characters of the elements of a given L-packet (or A-packet) Π of G to the ϵ-twisted character of the base change lift of Π.

It is a natural question to ask if a given instance of Langlands functoriality can be made explicit. That is, given classifications of the irreducible admissible representations of G and $\widetilde{G}$ in terms of compact open data, can one describe the correspondence of representations in terms of these classifications? This problem has been investigated for some instances of functoriality, e.g., the work of Bushnell-Henniart [3, 4, 5, 6], where the base change lifting for GL_n is given in terms of the classification of irreducible supercuspidal representations of GL_n via types due to Bushnell-Kutzko [7]. In a related vein, in conjunction with A. Raghuram [28], I show that the depth (a much coarser notion than the type) of a representation is preserved under the functorial correspondence [10, 34] between representations of $\mathrm{GL}_n(F)$ and certain of its inner forms.

Let $\underline{G}$ be the quasi-split unitary group $U(2,1)$ over F defined with respect to the quadratic extension E/F, and assume that the residue characteristic of F is not 2. In [1], J. Adler and I address the above question in the case of the base change lifting from $\underline{G}$ to $\underline{\widetilde{G}}$ for L-packets of depth-zero representations under the assumption that E/F is unramified. We completely describe the depth-zero L-packets of G and their base change liftings in terms of the classification of depth-zero representations via K-types by Moy-Prasad [32] . For example, we show that the stable supercuspidal representations are precisely those which arise via compact induction from representations of $U(2,1)(k_F)$ obtained by Deligne-Lusztig induction from characters of elliptic tori which correspond to cubic extensions of k_F. We also explicitly describe the base change lifts of such L-packets using the Shintani relation on the level of both p-adic and finite groups (see [20]), results on endoscopy due to Rogawski [35], and results on the action of $\widetilde{G} \rtimes \langle\varepsilon\rangle$ on its Bruhat-Tits building. Given a point x in the building $\mathcal{B}(\underline{G}, F)$ of $\underline{G}$ over F, let G_x denote the corresponding parahoric subgroup and G_x its finite reductive quotient. Define the groups $\widetilde{G}_x$ and $\widetilde{\mathsf{G}}_x$ similarly. (Note that $\mathcal{B}(\underline{\widetilde{G}}, F)$ contains $\mathcal{B}(\underline{G}, F)$.) If σ is a representation of G_x or $\widetilde{\mathsf{G}}_x$, let $\mathrm{infl}(\sigma)$ denote the inflation of σ to G_x or $\widetilde{G}_x$, respectively. The main theorem that we prove is the following.

Theorem 1. *Suppose Π is a depth-zero A-packet for G, let $\tilde{\pi}$ denote the base change lift of Π, and let $\pi \in \Pi$. Suppose $(G_x, \mathrm{infl}(\sigma))$ is a K-type contained in π. Then $\tilde{\pi}$ contains $(\widetilde{G}_x, \mathrm{infl}(\tilde{\sigma}))$, where $\tilde{\sigma}$ is the Shintani lift of σ from G_x to $\widetilde{\mathsf{G}}_x$.*

I am currently working on a similar explicit description of the base change lifts of depth zero L-packets when E/F is ramified. This would be of some significance as no work has yet been done on the problem of explicitly describing a base change lifting from a ramified group over a ramified extension. The ramified case may be made more complicated by the fact that for $x \in \mathcal{B}(\underline{G}, F)$, the group G_x need not be the group of ε-fixed points in $\widetilde{G}_x$. I plan to initiate a study of this phenomenon for general connected reductive groups over F.

I also expect that the arguments for stable representations of $U(2,1)$ will work for representations of unitary groups in a prime number of variables provided that certain properties of $U(2,1)$ hold more generally. Under these assumptions, I also believe that these methods will transfer to other reductive groups $\underline{G}$, as long as we have a handle on the characters of $\underline{G}$ in the finite and p-adic settings and on the action of $\widetilde{G}$ on its building.

The main result of [1] implies that, in the setting of unramified $U(2,1)$, the base change lifting on L-packets of depth zero representations behaves nicely with respect to K-types. A longer term goal would be to attempt to determine the compatibility between this lifting for L-packets of arbitrary depth and the construction of supercuspidals due to J.-K. Yu [40].

Finally, a related avenue of research will be to prove a formula for parabolically induced characters of a geometrically disconnected p-adic group, generalizing a formula of van Dijk [37] for connected groups. This is relevant to the above project in that the twisted character of a representation $\tilde{\pi}$ of $\widetilde{G}$ can be viewed as the restriction to $\widetilde{G}\varepsilon$ of the character of an extension of $\tilde{\pi}$ to $\widetilde{G} \rtimes \langle\varepsilon\rangle$.

2. Newforms

Another focus of my research is the development of a theory of conductors and newforms (or new vectors) for representations of certain p-adic groups that is compatible with the local factors of these representations. To introduce the main theme, we recall the following theorem of Casselman [8]. Let F be a non-Archimedean local field with ring of integers $\mathcal{O}_F$. Let $\mathcal{P}_F$ be the maximal ideal of $\mathcal{O}_F$, and let $q = |\mathcal{O}_F/\mathcal{P}_F|$. Let ψ be a nontrivial additive character of F such that the maximal fractional ideal on which ψ is trivial is $\mathcal{O}_F$.

Theorem 2 (Casselman)**.** *Let (π, V) be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(F)$. Let ω_π denote the central character of π. Set $\Gamma(0) = \mathrm{GL}_2(\mathcal{O})$ and for each positive integer m let $\Gamma(m)$ be the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $c \in \mathcal{P}_F^m$. Let $V_m = \left\{v \in V : \pi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) v = \omega_\pi(d)v \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(m)\right\}$.*

(i) *There exists a non-negative integer m such that $V_m \neq (0)$. If $c(\pi)$ denotes the least non-negative integer m with this property then the epsilon factor $\epsilon(s, \pi, \psi)$ of π is a constant multiple of $q^{-c(\pi)s}$.*

(ii) *For all $m \geq c(\pi)$ we have $\dim(V_m) = m - c(\pi) + 1$.*

The unique vector (up to scalars) in $V_{c(\pi)}$ is called the newform for π, and the assertion $\dim(V_{c(\pi)}) = 1$ is sometimes referred to as the multiplicity one theorem for newforms. This is closely related to the classical Atkin-Lehner theory of newforms for holomorphic cusp forms on the upper half plane [8]. Newforms play an important role in the theory of automorphic forms. For example, the local L-factor associated to π is exactly the zeta integral corresponding to a newform in π (see [18] for instance). Certain results analogous to those of Casselman for $\mathrm{GL}_2(F)$ are known for $\mathrm{GL}_n(F)$ [18, 30] and $\mathrm{GL}_2(D)$ where D is a p-adic division algebra [33]. In the case of $\mathrm{GL}_n(F)$, the determination of dimensions of fixed-spaces under filtration subgroups is of importance in local level raising arguments (as in [30]).

In [27], together with A. Raghuram, I begin the development of a theory of newforms for $\mathrm{SL}_2(F)$ and the quasi-split unramified unitary group $U(1,1)(F)$ when the residue characteristic is not 2. In particular, we prove an analogue of Theorem 2 for these groups. In addition, we relate the conductor of a representation π of one of these groups to various other invariants associated to π, e.g., depth (see [31]). Finally, we show that newforms for π are test vectors for a suitable Whittaker functional. A major complication in this theory which is not present in the case of $\mathrm{GL}_2(F)$ is the fact that representations of SL_2 need not be stable (i.e., two inequivalent irreducible representations may be conjugate by the adjoint group $\mathrm{PGL}_2(F)$). In particular, we need to consider not just irreducible representations but L-packets of representations.

As indicated above in the context of $\mathrm{GL}_2(F)$, the newforms of representations are closely connected to the local factors of those representations. In the case of $\mathrm{SL}_2(F)$, I am currently investigating the connection between the zeta integral (as defined by Gelbart-Jacquet [12]) of a newform for an irreducible representation π and the local L-factor associated to π. I expect (based on some preliminary calculations) that, as in the GL_2 case, such integrals are local L-factors. I also hope to prove a relationship between the epsilon factor of π and the conductor of π analogous to that in the case of GL_2. Another goal is the development of a theory of new forms for the more complicated quasi-split unramified unitary group $G = U(2,1)(F)$ that is compatible with local factors as defined Gelbart-Piatetski-Shapiro [13].

For possible global applications, I also plan to to complete the computation of conductors and newforms for $SL_2(F)$ in the case of residue characteristic 2. One such application might be the use of these newforms to define the field of rationality of a cuspidal automorphic representation π of SL_2 as Waldspurger does in [38] in the case of PGL_2. (This field is used in [39] to determine special values of certain L-functions attached to cuspidal automorphic representations of PGL_2.) This would be a longer term goal.

3. Algebraic Modular Forms and Hecke Algebras

Let $\underline{G}$ be a reductive group defined over $\mathbb{Q}$ such that $\underline{G}(\mathbb{R})$ is compact modulo the split center. Gross has developed a theory of algebraic modular forms for such $\underline{G}$. Let W be an algebraic representation of $\underline{G}$ and K an open compact subgroup of the group $\underline{G}(\widehat{\mathbb{Q}})$ of points of $\underline{G}$ over the finite adeles. Following [15, 16] we define the space of modular forms of weight W and level K on $\underline{G}$ to be the rational vector space

$$M(W,K) = \{F : G(\widehat{\mathbb{Q}})/K \to W(E) : F(\gamma g) = \gamma F(g), \text{ for all } \gamma \in G(\mathbb{Q})\}.$$

The close connection between these modular forms and automorphic representations of $\underline{G}$ is discussed in [26].

Suppose that the level K is a product of local factors $\prod_p K_p$. There is a natural action of each Hecke algebra $\mathcal{H}(\underline{G}(\mathbb{Q}_p), K_p)$ on $M(W,K)$. Denote by $\Gamma_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$, and let ${}^L\underline{G}$ be the L-group of $\underline{G}$, considered as a group over the algebraic closure $\mathbb{F}$ of a finite field. Gross [14] conjectures the existence of modular forms on $\underline{G}$ which correspond to certain representations $\rho : \Gamma_{\mathbb{Q}} \to {}^L G$ in the sense that the the images of certain Frobenius elements of $\Gamma_{\mathbb{Q}}$ under ρ are related to the actions of the local Hecke algebras on these forms. This is a generalization of Serre's conjectures concerning the existence of classical modular forms mod p attached to odd irreducible representations $\rho : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$.

In [26] with D. Pollack, I obtain a general formula for the decomposition of the supports of elements of $\mathcal{H}(\underline{G}(\mathbb{Q}_p), K_p)$ (which are unions of double cosets) into left cosets of K_p in the case where K_p is a hyperspecial or Iwahori subgroup of $\underline{G}(\mathbb{Q}_p)$. This decomposition is accomplished more generally in [24] for double cosets $P_1 \sigma P_2$ of subgroups P_1, P_2 of a semisimple split group G over a non-archimedean local field that contain a common Iwahori subgroup. In [26], we use this decomposition to carry out explicit calculations of the action of local Hecke algebras on spaces of algebraic modular forms on the compact form of G_2 and for several compact forms of PGSp_4. The data obtained in [26] is of interest from the point of view of number theory and of representation theory. For example, in light of Gross's conjectures, our data predicts the existence of a number field with Galois group $G_2(\mathbb{F}_5)$ that is ramified only at the prime 5. It also provided evidence for the existence of the symmetric cube lift from PGL_2 to PGSp_4 (which has now been established [21]). Although a great deal of data analogous to that in [26] is available for classical modular forms for GL_2, little previous information had been available for groups other than GL_2.

The conjectured relation between representations of $\Gamma_{\mathbb{Q}}$ and the actions of the algebras $\mathcal{H}(\underline{G}(\mathbb{Q}_p), K_p)$ on modular forms is defined via the Satake isomorphism at places p at which $\underline{G}(\mathbb{Q}_p)$ is split and the factor K_p of the level K is hyperspecial. The computations in [26] required an explicit knowledge of the matrix coefficients of the inverse of the Satake isomorphism. In the case where G is of adjoint type (so that the complex dual group $\widehat{\underline{G}}$ is simply connected), these coefficients are given by Kazhdan-Lusztig polynomials for the affine Weyl group of $\widehat{\underline{G}}$ [29]. Using a result of Kato [19] on these polynomials, I give a formula in [23] that explicitly computes them recursively with respect to the standard partial ordering on the character module of a maximal torus of $\widehat{\underline{G}}$.

For a reductive p-adic group G, one consequence of the Satake isomorphism is that the Hecke algebra $\mathcal{H}(G, K)$ is commutative for any hyperspecial subgroup K of G. Hence every simple $\mathcal{H}(G, K)$-module is one-dimensional. If I is an Iwahori subgroup of G, the structure of $\mathcal{H}(G, I)$ is more complicated, but well-known (see [17]). The simple $\mathcal{H}(G, I)$-modules have dimension at most $|W|$, where W is the relative Weyl group of G (see [9]). In [25], I generalize these results to arbitrary parahoric subgroups of G. I show that if $K \supset I$ is a parahoric subgroup of G and E is a simple module over $\mathcal{H}(G, K)$, then $\dim E \leq |W|/|W_K|$, where W_K is the subgroup of the affine Weyl group of G such that $K = IW_K I$. I also show this bound is sharp. The proof involves the explicit computation of the effects of generators of $\mathcal{H}(G, I)$ on I-fixed spaces in unramified principal series representations of G.

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