Derivatives

• Lectures #10-12:
  • Part V: Option pricing
    » Determinants of an Option’s Premium
    » Black-Scholes formula
    » Intro to Binomial Trees & Risk Neutral Valuation

• Lectures #11-13:
  • Part VI: Valuing Options in Practice
    » Binomial Trees & Risk-Neutral Option Pricing
    » Black-Scholes extensions

Part VI: Valuing Options in Practice

Practical Binomial Option Pricing

• Fundamentals
  • What? Why? How?

• Underlying Price Movements
  • Binomial trees

• Option Pricing
  • 1. no dividends
  • 2. continuous dividends
  • 3. discrete, known dividends

Binomial Option Pricing

• Basic idea
  • approximate the movements in an asset’s price
    » by discretizing the underlying’s price movements
    » to simplify the pricing of derivatives on the asset

• Realistic?
  • so far
    » 3-month or 1-year intervals
  • in practice
    » divide option’s life span into 30+ periods (ideally: 100+)
    » yields $2^{30} = \text{1 billion}$ possible price paths

Binomial Trees

• Asset Price Movements
  • divide option life ($t$ to $T$) into small intervals $\Delta t$
  • in each interval of time, assume asset price can move $UP \\downarrow$
    by a proportional amount $u$
  • move $DOWN \\uparrow$
    by a proportional amount $d$

Binomial Trees 2

• Moves in time interval $\Delta t$ ($H7$ Fig. 19.1; $H8$ Fig. 20.1)

• Derivatives can be “risk-neutrally” priced
  • expected return of all securities = risk-free rate
  • discounting of all cash-flows is done at risk-free rate
  • calls, puts, stocks, etc.
Tree Parameters

• What?
  • $p$, $u$, & $d$

• How?
  • tree must give correct values
  • for the mean & standard deviation
  • of the stock price changes
  • in a risk-neutral world (why?)

• Simplification
  • assume that $u = 1/d$

Tree Parameters 2

• 1. Nondividend Paying Stock

• Situation
  • need to find $u$, $p$ and $d$
  • find 3 equations with 3 unknowns
    • mean, variance, simplification

• a. mean of the stock price:
  • expected stock price: $pS_u + (1-p)S_d$
  • risk-neutral value: $Se^{\Delta t}$
  • hence (Eq. 20.1): $S e^{\Delta t} = pS_u + (1-p)S_d$

Tree Parameters 3

• b. standard deviation of the stock price:
  • variance: $pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2$
  • risk-neutral value: $S^2 \sigma^2 \Delta t$
  • hence: $S^2 \sigma^2 \Delta t = pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2$

• c. simplification
  • assume that $u = 1/d$

Tree Parameters 4

• a & b & c: approximate solution
  • if $\Delta t$ is small, then (Ch. 17, H6; Ch. 19, H7; Ch. 20, H8)
  
  $u = e^{\sigma \sqrt{\Delta t}}$  \hspace{1cm} (20.5)
  $d = e^{-\sigma \sqrt{\Delta t}}$  \hspace{1cm} (20.6)
  $p = \frac{a-d}{u-d} = \text{risk-neutral probability}$  \hspace{1cm} (20.4)
  $a = e^{\gamma \Delta t} = \text{growth factor}$  \hspace{1cm} (20.7)

Tree Parameters 5

• Full (Recombining) Tree
  (Fig.19.2 or 20.2)

Backwards Induction

• Idea
  • We know the value of the option
    • at the final nodes
  • Work back through the tree
    • using risk-neutral valuation
      • to calculate the value of the option at each node

• American vs. European options
  • American options
    • test for early exercise at each node (where appropriate)
Backward Induction 2 – Put Example

- Option parameters
  \( S = 50; \ X = 50; \ T = 5 \) months

- Other data
  \( \text{annualized risk-free rate} \quad r = 10\% \)
  \( \text{underlying annual std. dev.} \quad \sigma = 40\% \)

- Time parameters
  \( T = 5 \) months = \( \frac{5}{12} = 0.4167 \)
  \( \Delta t = 1 \) month = \( \frac{1}{12} = 0.0833 \)

Backward Induction 3 – Put Example

- Solution
  - parameters imply
    \( u = 1.1224; \ d = 0.8909; \ a = 1.0084; \ p = 0.5076 \)
  - in practice
    » solve tree manually (Fig. H7-19.2 or H8 20.2)
    » or use software
      - example: DerivaGem (Fig. 19.3 or 20.3)

Backward Induction 4 – Put Example

Fig. 20.3

Tree Parameters 6

- \(2.\) Dividend Paying Stock (continuous time)
  - dividend yield
    \( \rightarrow q \) (continuously compounded rate)
  - payout consequence
    » underlying price grows more slowly
    - as dividends are being paid out
  - risk-neutral valuation
    » must reflect lower growth rate of underlying price

Tree Parameters 7

- Situation
  - need to find \( u, p \) and \( d \)
  - find 3 equations with 3 unknowns
    \( \rightarrow \) mean, variance, simplification

- \( a. \) mean of the stock price:
  - expected stock price: \( pS^u + (1-p)S^d \)
  - risk-neutral value: \( S \cdot e^{r \Delta t} \)
  - hence (Eq. 19.1 or 20.1):
    \( S \cdot e^{r \Delta t} = pS^u + (1-p)S^d \)

Tree Parameters 8

- \(b.\) standard deviation of the stock price:
  - variance:
    \( pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2 \)
  - risk-neutral value:
    \( S \cdot \sigma^2 \Delta t \)
  - hence:
    \( S \cdot \sigma^2 \Delta t = pS^2u^2 + (1-p)S^2d^2 - S^2[pu + (1-p)d]^2 \)

- \(c.\) simplification
  - assume that \( u = 1/d \)
Tree Parameters 9

- $a$, $b$, and $c$: approximate solution
  - if $\Delta t$ is small, then (Ch. 20 in H8, Ch. 19 in H7)

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} \\
    d &= e^{-\sigma \sqrt{\Delta t}} \\
    p &= \frac{a - d}{u - d} \\
    a &= e^{(r-q) \Delta t}
\end{align*}
\]

Tree Parameters 10

- Relevance of the continuous-payout case
  - Analogy
    - treatment similar to Black-Scholes
  - Cases
    - stock index option
      - $q$: dividend yield on the index
    - foreign currency option
      - $q$: foreign risk-free rate $= r^*$
    - futures contracts option
      - $q = r$
      - why? ensures expected growth of $F$ in a R-N world is 0

Tree Parameters 11

- Examples of the continuous-payout case
  - DerivaGem software
    - e.g., importance of dividends for early exercise
    - IBM is currently trading at $S_0 = 86.50$
      - annualized interest rates are currently around $r = 1.75\%$
      - the annual stock return volatility is about $\sigma = 21\%$
      - strike $X = 90$: should you exercise an IBM call early?
        - IBM’s dividend yield is currently about $q = 2.61\%$
        - P.22: American call; p.23: European call

Tree Parameters 12

- Strike price = $X$
- Discount factor per step = $1.0007$
- Time step, $\Delta t = 1.0005$ years, 30.42 days
- Growth factor per step, $a = 1.0015$
- Probability of up move, $p = 0.4929$
- Up step size, $u = 1.0025$
- Down step size, $d = 0.9941$

Tree Parameters 13

- Strike price = $X$
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Tree Parameters 14

- 3. Dividend Paying Stock (yield known)
  - Problem
    - the dividend is paid once (or a few times)
    - during the life of the option
  - Solution
    - similar to case 2 (continuously paid dividends)
    - intuition
      - once the dividend has been paid
      - the tree recombines (Fig. 17.7 in H6, Fig. 19.7 in H7)
Tree Parameters 13

- 4. Dividend Paying Stock (value known)

Problem
- tree does not recombine

Solution
- draw an initial tree (uncertain component)
  - for the stock price less the present value of the dividends
- create the final tree (add certain component)
  - by adding the present value of the dividends at each node

Tree Parameters 14

- Ex-dividend date \( = \tau \) (Figs. 19.8-9 or 20.7-8)
  - tree step
    - \( i = 1, 2, \ldots, N \)
    - \( N \Delta t = T \)
  - Uncertain component's value at time \( i \Delta t \)
    - \( S^* = S \)
    - when \( i \Delta t > \tau \) (i.e., ex-dividend)
    - \( S^* = S - D^* \exp[-r(\tau - i \Delta t)] \)
    - when \( i \Delta t \leq \tau \) (i.e., cum-dividend)

Tree Parameters 15

- 4. IBM, no div.

Tree Parameters 16

- 4. June div. = 56c

Extensions

- Control-variate techniques
  - why?
  - when? Black-Scholes is OK
- Interest rates
  - in Black-Scholes, theoretical problem
  - here, simple solution (why?)
- Extra lecture
  - interest rate derivatives
Control-Variate Technique for American Options

• Use the same tree
  • to calculate the value of
    » American option, $f_A$ and corresponding European option, $f_E$
  • Let $f_{BS}$ = Black-Scholes price of the same option.
    » price of the American option can then be adjusted
    to $f_A + f_{BS} - f_E$

• Underlying assumption
  • "tree-errors" are the same
  • for European and American options

Control-Variate Technique for American Options 2

• Use the same tree
  • to calculate the value of
    » American option, $f_A = \$1.63$
    » and corresponding European option, $f_E = \$1.50$
  • Let $f_{BS}$ = B&S price of the same option = $\$1.52$
    » price of the American option can then be adjusted
    to $f_A + f_{BS} - f_E = \$1.63 + (1.52-1.50) = \$1.65$

• Underlying assumption
  • "tree-errors" are the same
  • for European and American options

Time-Varying Interest Rates

• Allow for interest rates to vary over time

• Before, $
p = \frac{a - d}{u - d}$
  $a = e^{rt}$

• Now, $p(t) = \frac{a(t) - d}{u - d}$
  $a(t) = e^{rt(t)}$