Derivatives & Risk Management

- Lectures 12-13:
  - Part VII: Option pricing & Continuous-time Finance
    - Modeling Asset Prices in Continuous Time
  - Part VIII: Option pricing – Advanced Topics
    - Derivation of the Black-Scholes Formula
    - Option Pricing w/ varying risk-free rate
- Also in this Lecture (14):
  - Part IX: Option Risk Management
    - The "Greeks"

Managing Market Risk

- Stochastic Process & Price Modeling
  - continuous time
  - continuous variables
- Markov Processes
  - market efficiency
- Wiener Processes
- Itô Processes & Itô’s Lemma

Example

- A FI has SOLD for $300,000 a European call option on 100,000 shares of a non-dividend paying stock:
  \[ S = 49 \quad X = 50 \quad r = 5\% \]
  \[ \sigma = 20\% \quad T-t = 20 \text{ weeks} \quad \mu = 13\% \]
- The Black-Scholes value of the option is $240,000
- How does the FI hedge its risk?

Example 2: Naked & Covered Positions

Naked position
- Take NO action

Covered position
- Buy 100,000 shares today

Both strategies leave the FI exposed to significant risk, e.g. \( S_T = 40, S_T = 60 \).

Example 3: Stop-Loss Strategy?

This involves:
- Fully covering the option as soon as it moves in the money
- Staying naked the rest of the time

This “deceptively simple” hedging strategy does NOT work well! Underlying must be bought slightly above \( X \) and sold slightly below \( X \).
Delta Hedging

• Idea:
  - Use the Black-Scholes insight
  - to construct a risk-free portfolio
    • buying delta shares
    • for every option which is written.

Delta Hedging 2: Definition

• Delta ($\Delta$) is the rate of change of the option price with respect to the underlying

Delta Hedging 3: Formulas

• The delta of a European call on a stock is $N(d_1)$
• The delta of a European put is $-N(-d_1) = [N(d_1) - 1]$
• The delta of a European call on a stock paying dividends at rate $q$ is $N(d_1)e^{-q(T-t)}$
• The delta of a European put is $e^{-q(T-t)}[N(d_1) - 1]$

Delta Hedging 4: Plots

• See Figure 17.3 or 18.3 (call) and 17.4 or 18.4 (put)
  • Delta of call always between zero and 1.
  • Delta of put always between -1 and 0.
• See also Excel spreadsheets
  • DerivaGem
• Delta changes most dramatically when $S = X$
• As $T-t$ gets small:
  • ITM deltas go up
  • OTM deltas go down (why?)
Delta of Call Options

0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10

0.0 0.2 0.4 0.6 0.8 1.0 1.2

Time to Maturity

ITM ATM OTM

Delta Hedging 5: Hedging

• This involves maintaining a delta neutral portfolio [see TABLES 17.2-3 (H7&7) & 18.2-3 (H8)]
• The hedge position must be frequently rebalanced
• Delta hedging a written option involves a “BUY high, SELL low” trading rule
  This is the cost of the hedge (in addition to transactions costs)

Example: Table 17.2-3 or 18.2-3, Wk 9

• B-S option value initially is $240,000. In week 9 the value is $414,500, so the FI has lost $174,500 on the option.
• The FI has also borrowed $1,442,700 in cash.
• But the shares have gone from $2,557,800 to $4,171,100 during the nine weeks.
• The net effect is a loss of $3,900
• When you rebalance infinitely often, the loss will be zero.

Using Futures to Hedge Options

• Futures could have different maturity, $T^*$
  $F = S \exp(r(T^*-t))$
• For hedging
  $\Delta S = \Delta F \exp(-r(T^*-t))$
• The required futures position is therefore
  $H_F = \exp(-r(T^*-t)) H_A$
  where $H_A = \text{req'd position in the underlying (S)}$
• Use ($r-q$) for $r$ when dividends are paid

Delta of a Portfolio

• The delta of a portfolio of options is just the weighted sum of the individual deltas (why?)
  $\Delta = \sum_{i=1}^{n} w_i \Delta_i$
• The weights $w_i$ equal the number of underlying option contracts written.

Theta

• Theta ($\theta$) of a derivative (or portfolio of derivatives) is the rate of change of its value with respect to the passage of time
  $\theta = \frac{\partial f}{\partial t}$
• See Figure 15.5 (H7; 18.5 in H8) for the variation of $\theta$ with respect to the stock price for a European call
• We do not hedge against the passage of time!
  • BUT, $\theta$ is useful to describe relationships between other hedge parameters.
Gamma

- **Gamma** ($\Gamma$) is the rate of change of delta ($\Delta$) with respect to the price of the underlying
  \[ \Gamma = \frac{\delta^2 f}{\delta S^2} \]

- Gamma indicates how often we should rebalance.
- See Figure 15.9 (H7; 18.9 in H8) for the variation of $\Gamma$ with respect to the stock price for a call or put option.

Gamma 2: Why do we need it?

- Idea: Gamma Addresses Delta Hedging Errors Caused By Curvature

Gamma 3: Interpretation

- For a delta neutral portfolio (e.g. short 1 call and long $\Delta$ shares), a Taylor series expansion series shows that $\Delta \Pi = 0 \Delta t + \frac{1}{2} \Gamma \Delta S^2$

Gamma 4: Making a Portfolio Gamma-Neutral

- As stocks and futures have zero Gamma, you need to buy/sell traded options to change Gamma
- Gamma of delta neutral portfolio plus traded option is $\Gamma + \omega_i \Gamma_T$, so set $\omega_i = -\Gamma / \Gamma_T$
- Changing traded option position changes delta, so adjust for this by changing stock position.

Gamma 5: Example

- Suppose a portfolio is delta neutral and has a gamma of -3,000. The delta and gamma of a particular traded option is 0.62 and 1.50 respectively.
- Make portfolio gamma neutral by buying 3,000/1.5 = 2,000 options.
- This changes delta from 0 to 0.62*2,000 = 1,240. So sell 1,240 shares of underlying to regain delta neutrality.
Gamma 6: Formulae

- For a European call or put w/ dividends
  \[ \Gamma = \frac{N'(d_1)e^{-qt}}{S\sigma \sqrt{T-t}} \]
  - Option on FX: \( q = r_f \)
  - Option on Future: \( S = F, \) and \( q = r. \)
- \( N'(d) \) is bell-curve: \( N'(d) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d^2}{2}\right) \)

Vega

- Vega (\( V \)) is the rate of change of a derivatives portfolio with respect to volatility
  \[ V = \frac{\partial f}{\partial \sigma} \]
- See Figure 15.11 for the variation of \( V \) with respect to the stock price for a call or put option
- In reality, volatility does seem to change over time so we might want to hedge that risk away.

Vega 2: Formulae

- For a European call or put w/ dividends
  \[ \text{Vega} = S\sqrt{\bar{T}} - \bar{t}N'(d_1)e^{-qt(T-t)} \]
  - Option on FX: \( q = r_f \)
  - Option on Future: \( S = F, \) and \( q = r. \)
  - Vegas from stochastic volatility models are similar to the our B-S approximate Vega.

Relationship Among Delta, Gamma, & Theta

Recall the B-S differential equation
\[ \frac{\partial f}{\partial t} + rSF \frac{\partial f}{\partial S} + 0.5\sigma^2S^2 \frac{\partial^2 f}{\partial S^2} = rf \]  (13.16)  p. 288

For a non-dividend paying stock
\[ \theta = rS \Delta + \frac{1}{2\sigma^2S^2}T - rf \]
Thus, when delta is 0, and theta is large and positive, gamma tends to be negative.

Vega 3:
Making a Portfolio Vega-Neutral

- As stocks and futures have zero Vega, you need to buy/sell traded option to change Vega
- Vega of delta neutral portfolio plus traded option is \( V + w_T V_T, \) so set \( w_T = -V / V_T \)
- Changing the traded option position changes delta, so adjust for this by changing stock position
Gamma- and Vega-Neutral: Example

- Consider a delta neutral portfolio with a gamma = -5,000; vega = -8,000.
- Suppose a traded option has gamma = 0.5; vega = 2.0; delta = 0.6.
- Suppose another traded option has gamma = 0.8; vega = 1.2; delta = 0.5.

Example [cont.]

- For joint neutrality, we need
  \[-5,000 + 0.5w_1 + 0.8w_2 = 0,\]
  \[-8,000 + 2.0w_1 + 1.2w_2 = 0\]
- Solving, yields \( w_1 = 400, \) and \( w_2 = 6,000.\)
- Now, delta is 400*0.6 + 6,000*0.5 = 3,240.
- Thus, sell 3,240 units of the underlying to reestablish delta neutrality.

Managing Delta, Gamma, & Vega

- \( \Delta \) can be changed by taking a position in the underlying.
- To adjust \( \Gamma \) & \( \nu \) it is necessary to take a position in an option or other derivative.

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate:
  \[\rho = \frac{\partial f}{\partial r}\]
- For currency options there are 2 rhos.

Scenario Analysis

- Scenario analysis & the calculation of value at risk (VaR) is a complement to \( \Delta, \Gamma, \nu, \) etc.
- Typical VaR question: What loss level are we 99% certain will not be exceeded over the next 10 days?

Scenario Analysis [cont.]

- Delta and Gamma are used to calculated VaR’s of portfolios including options.
- Stress testing involves checking scenarios with extreme values for the variables (e.g. FX rate and volatility).
- Monte Carlo Simulation involves generating thousands of scenarios and compute the histogram of returns across simulations. VaR can then be found as the e.g. 5th percentile.
Hedging vs
Creation of an Option Synthetically

• When we are hedging we take positions that offset $\Delta$, $\Gamma$, $\nu$, etc.
• When we create an option synthetically we take positions that match $\Delta$, $\Gamma$, & $\nu$

Portfolio Insurance

• In October of 1987 many portfolio managers attempted to create put options on their portfolios by matching $\Delta$
• This involves initially SELLING enough of the portfolio (or of index futures) to match the $\Delta$ of the put option
• Might work well if options markets are not very liquid and/or desired strikes are not traded.

Portfolio Insurance [cont.]

• As the value of the portfolio increases, the $\Delta$ of the put becomes less negative & the position in the portfolio is increased
• As the value of the portfolio decreases, the $\Delta$ of the put becomes more negative & more of the portfolio must be SOLD
• The strategy did NOT work well on October 19, 1987...