

Note on Floaters and Swaps

1 Valuation of Floaters

A floater is nothing but a series of one period deposits that is rolled over until maturity with the interest being paid out instead of reinvested. Since its interest rate is set in advance at all the reset dates the fictitious one period deposit is valued at par at the beginning of each period: setting the return (coupon rate) on a deposit (bond) means that the instrument has to trade at par just after its inception for this period. Take a bond where YTM = c, the coupon rate: it has to sell at par, otherwise there are arbitrage opportunities. This simple observation yields a surprising but powerful method for pricing floaters and a swap’s floating leg, in which the reset rates do not even show up. Although the explicit absence of reset rates in the calculation is initially quite mystifying it simply reflects both the mathematics of floaters and implied forward rates, and the arbitrage logic underlying our discussions.

Since a floater’s coupon is always (re)set at par by definition the preceding discussion implies that it has to be priced at par in its last period. Given that it is priced at par in its last period it has to be valued at par the period before, too, and so on where in our previous method the following periods’ reset rates are “estimated” as implied forward rates. Therefore, the floater is priced at par conditional on knowing the reset rates! As it turns out the lack of foresight about reset rates is surprisingly not really a constraint: we can cut short our usual swap table procedure and directly to price the floating rate part by the following trick based on the par valuation observation. To see this consider reset date t = T − 1: the reset rate is known as rT so that the floater’s value at maturity T is 100 (1 + rT); discounting this expression back to t = T − 1 at the appropriate discount rate rT yields FRN_{T−1} = 100. Now, repeat the argument for t = T − 2 where one gets 100 (1 + r_{T−1}) which, similarly, needs to be discounted at the known rate r_{T−1} to give FRN_{T−2} = 100, etc. down to FRN_0 = 100.

The arbitrage argument is reflected by the appropriate algebra. From the replication and hedging arguments we know that the value of an FRN is calculate as the present value of its future cashflows. Since these cashflows can be hedged at any given time in the forward market their value must equal the return of forward deposits of appropriate timing and maturity. By the no-arbitrage condition, reset rates can thus be calculated as forward rates. Note that this is the replication argument: a floater can be synthetically replicated by a collection of forward contracts. One obtains that \( FRN_0 = N \left[ \sum_{t=1}^{T} d_t f_{t−1,t} + d_T \right] \) for notional value N, discount factors \( d_t \) and forward rates \( f_{t−1,t} \). But now, \( d_t = Z (0, t) \), i.e., the discount factor equals the price of a zero
paying $1 at maturity $t$; moreover, $f_{t-1,t} = \frac{Z(0,t-1)}{Z(0,t)} - 1$ by the definition of zero prices and spot rates as $Z(0,0) = 1$. As a result one has upon substituting for $d_t$ and $f_{t-1,t}$ by $Z(0,0) = 1$:

\[ FRN_0 = N \left[ \sum_{t=1}^{T} d_t f_{t-1,t} + d_T \right] = N \left[ \sum_{t=1}^{T} Z(0,t) \left( \frac{Z(0,t-1)}{Z(0,t)} - 1 \right) + Z(0,T) \right] \]

\[ = N \left[ \sum_{t=1}^{T} (Z(0,t-1) - Z(0,t)) + Z(0,T) \right] = N [1 - Z(0,t)] + Z(0,T) = N. \]  

2 Swap Hedging

Consider a plain-vanilla swap: the fixed-for-floating payment streams mean that it is a combination of long and short positions in floaters and bonds. Selling a swap means from the investment bank’s point of view that the bank receives fixed and pays floating. But this is equivalent to the cashflows of a long position in a bond and a short one in a floater. Hence, a swap can be synthetically replicated by a long position in a bond a short one in a floater. To hedge the exposure resulting from selling a swap then means that the investment bank has to reverse the positions. They would have to sell/issue a bond with coupon equal to the fixed rate and buy with the proceeds a floater indexed to the appropriate spot rate.

Hedging a long position in a swap once again requires to reverse the resulting positions. Having bought a swap means paying fixed and receiving floating for the swap dealer which is equivalent to having issued a bond (short) and invested in a floater (long). So, a dealer might cover the position by buying an appropriate amount of T-notes with the proceeds from selling issuing a floater indexed to the swap’s underlying spot rate. futures contracts (not necessarily the swap’s face value: remember basis risk and hedge ratio for futures?). If one hedges in the futures markets instead of the cash markets one might run into liquidity problems for longer swaps on the bond side.

On the floating side a further complication arises. For LIBOR swaps one would have to replicate the short position in the floater by a collection of short positions in Eurodollar futures contracts with maturities of $t = 1, \ldots, T$. However, the underlying contract is only a 3 months deposit so that one needs to string several of them together to get the yearly LIBOR deposit contracts. Unfortunately, this does not fully eliminate interest rate risk. one might incur timing mismatches and the market is quite illiquidity for delivery dates beyond 3 years.

3 Par Swaps and Swap Rates

As mentioned in class, there are two ways of pricing swaps. For a given fixed rate one could calculate the swap’s fair value or, alternatively, set the fixed rate so that the fair value is equal to 0. The first method corresponds to an up-front fee that would equate the present expected value of the floating cashflows to the present expected value of the fixed ones. In the alternative pricing method one solves for the fixed rate in terms of the other variables rather than the fee from floating and fixed rates. Put differently, the fixed rate is set in such a way so as to equate the floating with the
fixed leg in expected present value terms. This means that one has to solve for \( c = c_t = \text{fixed} \) in

\[
\sum_{t=1}^{T} \frac{c_t}{(1 + r_t)^t} = \sum_{t=1}^{T} \frac{f_{t-1,t}}{(1 + r_t)^t} \Rightarrow c : \sum_{t=1}^{T} \frac{c - f_{t-1,t}}{(1 + r_t)^t} = 0.
\]

In swap terminology, \( c \) is the swap rate: it is the fixed rate that has the same present value as the forward (reset) rates. Swaps where the fixed rate \( c \) is set so that the expected present value of floating and fixed payments are equalized are called par swaps.

It is quite straightforward to solve \( c : \sum_{t=1}^{T} \frac{c - f_{t-1,t}}{(1 + r_t)^t} = 0 \) for the swap rate that equates the return of the floating and the fixed leg where \( f_{t-1,t} \) is the floating rate gross of the spread over the relevant index:

\[
c = \left[ \sum_{t=1}^{T} \frac{1}{(1 + r_t)^t} \right]^{-1} \sum_{t=1}^{T} \frac{f_{t-1,t}}{(1 + r_t)^t} = \left[ \sum_{t=1}^{T} d_t \right]^{-1} \sum_{t=1}^{T} d_t f_{t-1,t} \quad (2)
\]

Since \( P(0,t) = \frac{1}{(1 + r_t)^t} = d_t \) is the price of a zero-coupon bond or the discount factor one can see how swaps are a portfolio of zero bonds and interest rate forwards. From the above formula one has that the fixed leg’s present value is

\[
V_c(0) = N c \sum_{t=1}^{T} P(0,t) \quad (3)
\]

for notional principal \( N \). This expression is then equated to the present value of the floating leg calculated either from our usual forward rate argument or from the following observation on floating rate notes.

Recall that a plain-vanilla swap is an appropriate combination of a short and long position in a floater and a coupon bond. So, view the swap’s floating leg in terms of a floater that is priced at par since the variable coupon rate is determined at the beginning of each reset period. As a result, its value \( V_F(0) \) is the notional principal \( N \): \( V_F(0) = N \) ! However, contrary to a floater the interest rate swap’s principal does not change hands. Hence, we need to subtract from \( V_F(0) \) the notional principal’s present value to get the value of the floating leg as the net present value of the notional floater. Let \( T \) be the swaps maturity and \( Z(0,T) \) the appropriate floating (e.g., LIBOR) discount factor derived from the floater’s underlying discount yield curve:

\[
V_F(0) = N - Z(0,T) N = N (1 - Z(0,T)) \quad (4)
\]

Put differently, a bond trades at par \( N \) for a set interest rate as is the case for a floater; however, since there is no principal payment in a swap one needs to deduct the present value of the principal in the form of \( Z(0,T) N \) from par to get the floating leg’s present value. But this is precisely the value of a floater by the above argument: \( V_F(0) = FRN_0 ! \) This also follows from (1) where it is shown that \( \sum_{t=1}^{T} d_t f_{t-1,t} = 1 - Z(0,T) \).

Setting \( V_c(0) = V_F(0) \) then yields from the preceding two equations the swap rate as

\[
c = \left[ \sum_{t=1}^{T} P(0,t) \right]^{-1} (1 - Z(0,T))
\]
while the swap’s value (up-front fee) is given as

$$\gamma_0 = V_c(0) - V_F(0) = N \left[ \sum_{t=1}^{T} P(0,t) - (1 - Z(0,T)) \right].$$

Initially, interest rate swaps were settled by exchanging the fair value as an up-front fee for a given fixed rate; very soon after their inception in 1982, however, swap dealers started to price them in terms of fixed rates. This is both more natural in view of FI market conventions and also more practical. Swap rates lend themselves to bid-ask quotation (in ascending order!!) and decrease default risk with respect to initial payments (one less payment to make). Moreover, they can be quoted with respect to the current spot rate, e.g. as 6M TB + 30 - 35 where the first swap rate is the bid rate (T-bill plus 30 bpts) at which the bank stays ready to buy a swap (pay fixed, receive floating), and the second the ask rate (T-bill plus 35 bpts) at which the bank sells swaps (receives fixed, pays floating). Notice how this agrees with the notion that customers always get the worse deal regarding bid-ask spread: they receive a lower fixed rate than they have to pay. Now, why would this be the case one wonders??

4 Foreign Currency Swaps

Foreign currency swaps are the oldest type of swap. They evolved in the golden times of fixed exchange rates as a means between central bank to counterbalance adverse swings in foreign currency reserves. Contrary to interest rate swaps, two parties borrow in different currencies and then agree to swap both principal and future loan payments. There are four basic variants that arise from the combination of fixed-for-floating and the two currencies. Take a plain vanilla currency swap between parties $A$ and $B$ and currencies GBP and USD. $A$ borrowed in Sterling but needs dollars, $B$ borrowed in dollar but needs Sterling. Hence, the two parties agree to swap principals and liabilities at $t = 0$:

$$\begin{align*}
\text{Counterparty} & \quad \text{GBP 50m} & \quad \text{USD 80.5m} & \quad \text{Counterparty} & \quad r_{USD\%} \quad \text{(USD)} \\
7.25\% \quad \text{(GBP)} & \quad A & \quad \text{GBP 50m} & \quad \text{USD 80.5m} & \quad B & \quad 7.25\% \quad \text{(GBP)} \\
\end{align*}$$

At maturity, the principals are returned to the respective creditors:

$$\begin{align*}
\text{Counterparty} & \quad \text{USD 80.5m} & \quad \text{Counterparty} & \quad \text{GBP 50m} \\
A & \quad \text{GBP 50m} & \quad B
\end{align*}$$

An FX swap’s valuation is straightforward: just use the up-front fee method. Incidentally, FX swaps preceded and inspired interest rate swaps, which explains why the latter were initially valued by the up-front fee approach. The required data is obvious: the spot FX rate, the respective loans’ specifications and the two yield curves. Let $S_{GBP/USD}(0)$ be the current price of 1 USD in terms
of GBP and \( V_{FX} (0) \) the present value of the loan in FX using the appropriate FX yield curve. Then the value of the preceding FX swap is

\[
V_{GBP/USD} (0) = V_{GBP} (0) - S_{GBP/USD} (0) V_{USD} (0) \tag{5}
\]

where the GBP party receives or pays an appropriate fee.

In reality, FX swaps are quoted in terms of swap rates, too. Moreover, principals are adjusted so that \( V_{GBP/USD} (0) = 0 \) at the inception of the swap which is a prerequisite for the computation of FX swap rates. Also notice that the maturities of the two loans are assumed to coincide but that the loans could carry either a fixed or a variable rate. To calculate the relevant swap rate one proceeds in complete analogy with equations (3) and (4). The value of the swap’s legs, in turn, yield the swap rate or its value. A small difficulty arises that one could look from either currency side at the swap. In the preceding example the swap fee computation proceeds from the British point of view in the formula given: either you turn everything around and look at it from an American point of view using the USD/GBP exchange rate \( S_{USD/GBP} (0) \) (this is, incidentally, how Sterling is quoted) or you use the fact that \( S_{GBP/USD} (0) = \left[ S_{USD/GBP} (0) \right]^{-1} \), a well known no-arbitrage condition in FX calculus.

The USD swap rate is the rate on a fictitious USD loan that will equalize the present value of the two swap legs including exchange of principal. The Sterling leg present value in USD is \( S_{\$} / £ (0) \cdot V_{GBP} (0) = S_{\$} / £ (0) N_£ \left[ \frac{c_£}{2} \sum_{t=1}^{T} P_£ (0, t) + P_£ (0, T) \right] \) while the USD leg present value comes to \( V_{USD} (0) = N_\$ \left[ \frac{c_\$}{2} \sum_{t=1}^{T} P_\$ (0, t) + P_\$ (0, T) \right] \) so that from \( V_{USD} (0) = S_{\$} / £ (0) \cdot V_{GBP} (0) \) the USD swap rate \( c_\$ \) follows in analytic form as

\[
c_\$ = 2 \left\{ S_{\$} / £ (0) N_£ \left[ \frac{c_£}{2} \sum_{t=1}^{T} P_£ (0, t) + P_£ (0, T) \right] - P_\$ (0, T) \right\}.
\]

Note that the preceding expression is for fixed-for-fixed FX swaps with semi-annual payments. You can derive the analogous expressions for fixed-for-floating FX swaps by using the cashflow matching argument outlined above.