Note on Caps and Floors

Spot-rate options (different from “yield options” to distinguish them from bond options) are written on an underlying interest-rate index which pays the positive part of the difference between a strike rate the reference index. Hence, they are interest-derivatives in which the buyer has the right but not the obligation to receive payments at the end of each period depending on the difference between reference and strike rate.

1. **cap**: an interest rate cap is a series of European call options (“caplets”) on a specified reference rate derivative such as LIBOR in which the buyer receives payments at the end of each period in which the interest rate exceeds the agreed strike rate; e.g., an agreement to receive a payment for each month the LIBOR rate exceeds 5%.

2. **floor**: an interest rate floor is a series of European put options (“floorlets”) on a specified reference rate, usually LIBOR in which the buyer receives payments if on the maturity of any of the floorlets, the reference rate is below the agreed strike price of the floor; e.g., an agreement to receive a payment for each month the LIBOR rate falls below 5%.

A cap with maturity $T$, strike rate $r_K$, and annual payment frequency $n$ is a security paying at times $T_1, T_2, ..., T_m = T$ with $T_{i+1} = T_i + \Delta$ and $\Delta = \frac{1}{n}$ (duration of payment period as an annual fraction) has the following cash flow at $T_i$ for notional underlying amount $N$:

\[ C(T_i) = \Delta \cdot N \cdot \max [r_n(T_{i-1}, T_i) - r_K, 0] \]

The analogous cash flows for a floor are

\[ F(T_i) = \Delta \cdot N \cdot \max [r_K - r_n(T_{i-1}, T_i), 0] \]

To value spot-rate options, one uses either lattice or closed-form models such as Black’s framework although increasingly Wall Street now relies on numerical simulations. A quick and dirty way to price caps and floors is via interest rate lattices. Here are the details for a worked example (see also the assignment). Suppose we have the following BDT spot-rate lattice:

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>6.000</td>
<td>6.3000</td>
<td>6.6150</td>
</tr>
<tr>
<td>$r_u$</td>
<td></td>
<td>6.57143</td>
<td></td>
</tr>
<tr>
<td>$r_d$</td>
<td></td>
<td>5.7143</td>
<td></td>
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</tbody>
</table>
To price a caplet means that we have to value an at-the-money European call on the 1Y LIBOR rate expiring in one year’s time at \( t = 1 \). First, notice that the strike rate is \( r_K = 0.0600 \) so that the payoff diagram is

\[
\begin{array}{ccc}
t = 0 & t = 1 & t = 2 \\
 \begin{array}{ll}
r_u = 6.3000 & \max \{6.3000 - 6.0000, 0\} = 0.30 \\
r_0 = 6.000 & \\
r_d = 5.7143 & \max \{5.7143 - 6.0000, 0\} = 0.00 \\
\end{array}
\end{array}
\]

Since it is a rate and not a yield option you directly use the above lattice. From \( c_s(1) = e^{-r_s} [r_s - r_K]^+ \), \( s \in \{u, d\} \) where \( f(x) = [x]^+ = \max \{x, 0\} \), i.e., is just short hand for “positive part,” one obtains the caplet’s price as

\[
c(0) = e^{-r_0} \frac{1}{2} [c_u(1) + c_d(1)] = e^{-0.06} \frac{1}{2} [e^{-0.063} (6.3 - 6.0) + 0] \\
= (0.47088) [0.281683] = 0.13263\%
\]
or 13.263 bpts under the BDT risk-neutral probabilities \( q = \frac{1}{2} = 1 - q \). If you interpreted the question as pertaining to LIBOR yield options you would have to extract a yield lattice from the spot rate lattice using the local expectations hypothesis. However, since it is a one year option on a one year yield you should get exactly the same answer unless you mix continuous and discrete compound interest!

Notice that there is one crucial difference between spot rate options such as caps and floors and yield options: discounting the final payoff back to the expiration date. Spot rate options corresponds to options on fictitious coupons that need to adjusted for the fact that they are paid at the end of spot rate’s time period, i.e., one year after the option’s expiration in the present case. Hence, one has the additional \( e^{-r_s} \) factor in the payoff expression. Yield options, on the other hand, are already adjusted for the time value of money (like futures prices) since their derivation is in terms of the underlying zero prices that represent the time value of money by their very definition. If one does not adjust the cap’s payoffs for the one year payment horizon one gets an error of almost 1 bpt:

\[
e^{-0.06} \frac{1}{2} [(6.3 - 6.0) + 0] = (0.47088) [0.3] = 0.14126\%.
\]

Now, let’s price a floorlet which is simply a European put option on the spot rate. It is calculated in a completely analogous fashion for payoffs \( p_s(1) = e^{-r_s} [r_K - r_s]^+ \), \( s \in \{u, d\} \) as 12.705 bpts from

\[
p(0) = e^{-r_0} \frac{1}{2} [p_u(1) + p_d(1)] = e^{-0.06} \frac{1}{2} [0 + e^{-0.057143} (6.0 - 5.7143)] \\
= (0.47088) [0.26983] = 0.12705.
\]

Omitting the discounting in \( p_s(1) \) induces an error of approximately 0.7 bpts:

\[
p(0) = e^{-0.06} \frac{1}{2} [0 + e^{-0.057143} (6.0 - 5.7143)] = (0.47088) [0.2857] = 0.13453.
\]
Can you price the corresponding two year caplets and floorlets and put everything together to find the two year cap and floor prices?