§3.4: 1. If $m$ is any integer, then $m(m + 1) = m^2 + m$ is the product of $m$ and its successor. That is to say, $m^2 + m$ is the product of two consecutive integers. The results of the preview activities verify that this product is even by examining the two possibilities: $m$ is odd, or $m$ is even. In both cases, the product of $m$ with $m + 1$ is divisible by 2.

2. We prove the following proposition.

**Proposition.** Suppose that $u$ is an odd integer; then the equation $x^2 + x - u = 0$ has no integer solutions.

**Proof.** Suppose for contradiction that there exists an integer solution $m$ of the equation

$$x^2 + x - u = 0,$$

where $u$ is an odd integer. We will examine the two cases: $m$ is odd and $m$ is even. If $m$ is odd, then there exists an integer $k$ such that

$$m = 2k + 1.$$

Since $m$ is a root of our equation, it must be that

$$(2k + 1)^2 + (2k + 1) - u = 0.$$

This implies that

$$4k^2 + 4k + 1 + 2k + 1 - u = 0.$$

This is clearly not possible, for $4k^2 + 10k + 2$ is even. Now suppose that $m$ is even. If so, then there exists an integer $k$ such that $m = 2k$; then

$$4k^2 + 2k - u = 0.$$

This is also impossible, for $4k^2 + 2k$ is even. \[\square\]

3. We prove the following.

**Proposition.** If $n$ is an odd integer, the $n = 4k + 1$ for some integer $k$ or $n = 4k + 3$ for some integer $k$. 


Proof. Suppose that \( n \) is odd; then there is an integer \( j \) such that \( n = 2j + 1 \). There are two possibilities for \( j \). Either \( j \) is even, or \( j \) is odd. If \( j \) is even, then there is an integer \( k \) such that \( j = 2k \). Thus

\[ n = 4k + 1. \]

On the other hand, if \( j \) is odd, there is an integer \( k \) such that \( j = 2k + 1 \). This means that

\[ n = 2(2k + 1) + 1. \]

This is equivalent to

\[ n = 4k + 3. \]

\( \square \)

6. (a) We prove the following. 

**Proposition.** If \( m \) and \( n \) are consecutive integers, then \( 4 \) divides \( m^2 + n^2 - 1 \).

**Proof.** Suppose that \( m \) and \( n \) are consecutive integers; we can assume that \( n = m + 1 \) in this situation. Then

\[ m^2 + n^2 - 1 = m^2 + (m+1)^2 - 1 = m^2 + m^2 + 2m + 1 - 1 = 2m^2 + 2m = 2m(m+1). \]

Now there are two cases: either \( m \) is even, or \( m \) is odd. If \( m \) is even, then \( 4 \) will divide \( 2m \), and therefore \( 2m(m+1) \). If \( m \) is odd, then \( m + 1 \) is even. \( 2 \) will divide \( 2m \), and \( 2 \) will also divide \( m + 1 \). It follows that \( 4 \) will divide \( 2m(m+1) \).

On the other hand, the converse is untrue. Let \( m = 1 \) and let \( n = 4 \). 

(b) We prove the following. 

**Proposition.** For all integers \( m, n \), if \( 4 \) divides \( m^2 - n^2 \), then either both \( m \) and \( n \) are even, or \( m \) and \( n \) are odd.

**Proof.** First suppose that both \( m \) and \( n \) are even. If so, then there exist integers \( j \) and \( k \) such that \( m = 2j \) and \( n = 2j \). So

\[ m^2 - n^2 = 4k^2 - 4j^2 = 4(k^2 - j^2). \]

This implies that \( 4 \) divides \( m^2 - n^2 \).

Now suppose that \( m \) and \( n \) are both odd. If so, then there exist integers \( k \) and \( j \) such that \( m = 2k + 1 \) and \( n = 2j + 1 \). So

\[ m^2 - n^2 = 4k^2 + 4k + 1 - 4j^2 - 4j - 1 = 4(k^2 + k - j^2 - j) \]

This means that \( 4 \) divides \( m^2 - n^2 \) in this case also. 

**Proposition.** For all integers \( m, n \), if \( 4 \) divides \( m^2 - n^2 \), then either both \( m \) and \( n \) are even, or \( m \) and \( n \) are odd.
Proof. Consider the contrapositive. Suppose that \( m \) is odd and \( n \) is even. If so, then there are two integers \( j \) and \( k \) such that \( m = 2j + 1 \) and \( n = 2k \). So

\[
m^2 - n^2 = (2j + 1)^2 - 4k^2 = 4j^2 + 4j + 1 - 4k^2.
\]

This is an odd number, so is not divisible by 4.

Suppose that \( m \) is even and \( n \) is odd. If so, then there are two integers \( j \) and \( k \) such that \( m = 2j \) and \( n = 2k + 1 \). So

\[
m^2 - n^2 = 4j^2 - 4k^2 - 4k - 1.
\]

This number is also odd, so it is not divisible by 4. \( \square \)

7. We prove

**Proposition.** If \( n \) is an odd integer, then \( 8 \mid n^2 - 1 \).

*Proof.* If \( n \) is odd, then there is an integer \( k \) such that \( n = 2k + 1 \). Then \( n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4(k^2 + k) \). By a result proved in class (and appearing elsewhere in this homework assignment), \( k^2 + k \) must be even, i.e., \( k^2 + k = 2j \) for some integer \( j \). Thus \( n^2 - 1 = 4(k^2 + k) = 4(2j) = 8j \), so \( 8 \mid n^2 - 1 \). \( \square \)

10. (b) We prove the following.

**Proposition.** For all real numbers \( x \) and \( y \), \( |xy| = |x||y| \).

*Proof.* There are several cases to consider. Clearly, if either \( x \) or \( y \) is equal to 0, then \( |xy| = |x||y| \). The equation is clearly true if \( x > 0 \) and \( y > 0 \), for \( |xy| = xy = |x||y| \) in this case. Suppose that \( x > 0 \) and \( y < 0 \). Then, \( |xy| = -xy \). On the other hand, \( |x| = x \) and \( |y| = -y \), so \( |x||y| = -xy \) as well. Similarly, the equation is true if \( x < 0 \) and \( y > 0 \). Finally, consider what happens when \( x < 0 \) and \( y < 0 \). In this case, the product \( xy > 0 \), so \( |xy| = xy \). On the other hand, \( |x| = -x \) and \( |y| = -y \). This implies that \( |x||y| = xy \) as well. \( \square \)

13. (a) This proof is incorrect, but the statement is true. It correctly deduces that \( an^3 + 2bn = 3 \), but then it incorrectly factors the left side as \( n(an^2 + b) \). Also, since \( n \) and \( an^2 + b \) are both integers, \( n > 0 \), and 3 is a prime number, it does follow (as in the proof) that one of \( n, an^2 + b \) has to equal 3 and the other has to equal 1. This proof only considers the case where \( n = 3 \) and \( an^2 + b = 1 \). The \( n = 1 \) needs to be considered, and doing so is analogous to the \( n = 3 \) case.

The proof should proceed as follows. Rewrite \( an^3 + 2bn = 3 \) as \( n(an^2 + 2b) = 3 \). Since \( n \) and \( an^2 + 2b \) are both integers whose product is 3, and since \( n \) is positive \( an^2 + 2b \) must be a positive. Since 3 is prime, there are only two possibilities: \( n = 3 \) or \( n = 1 \). If \( n = 3 \), then we have

\[
a(3)^3 + 2b(3) = 3,
\]
and dividing both sides by 3, we obtain

\[ 9a + 2b = 1. \]

If \( n = 1 \), then we have

\[ a + 2b = 3. \]

This completes the proof.

(b) This proof is incorrect, since it assumes the negation of the hypothesis of the statement. A correct proof by contradiction assumes the negation of the conclusion. See (a) for a correct proof.

§5.1: 1. (a) \( A = B \). It is evident that \( A \subseteq B \). If \( x \in B \), then \( x = 3, x = -3, x = 2, \) or \( x = -2 \). So, \( B \subseteq A \).

(b) Yes.

(c) No, the set \( C \) is not equal to the set \( D \). The set \( C \) is empty. The set \( D \) contains many, many real numbers.

(d) Yes; the empty set is a subset of any set.

(e) No, \( A \) contains negative numbers, e.g. \(-3\), but \( D \) doesn’t.

3. The following statements are true. \( A \subseteq B, A \subset B, A \neq B \). \( 5 \in C \). \( A \subseteq C, A \subset C, A \neq C \). \( \{1, 2\} \not\subseteq A, \{1, 2\} \neq A \). \( 4 \not\in B \). \( \text{card}(A) = \text{card}(D) \). \( A \in \mathcal{P}(A) \). \( \emptyset \not\subseteq A, \emptyset \subset A, \emptyset \neq A \). \( \{5\} \subseteq C, \{5\} \subset C, \{5\} \neq C \). \( \{1, 2\} \subseteq B, \{1, 2\} \subset B, \{1, 2\} \neq B \). \( \{3, 2, 1\} \subseteq D, \{3, 2, 1\} \subset D, \{3, 2, 1\} \neq D \). \( D \not\subseteq \emptyset, D \neq \emptyset \). \( \text{card}(A) < \text{card}(B), \text{card}(A) \neq \text{card}(B) \). \( A \in \mathcal{P}(B) \).

5. (a) It is not the case that \( \{a, b\} \subseteq \{a, c, d, e\} \). This is because there is an element \( b \in \{a, b\} \) which is not an element of \( \{a, c, d, e\} \).

(b) This is true. The set \( \{x \in \mathbb{Z} \mid x^2 < 5\} \) is the set \( \{-2, -1, 0, 1, 2\} \). The even members of this set are \( \{-2, 0, 2\} \).

(c) This is true. The empty set is a subset of any set.

(d) This is not true. The set \( \{a\} \) is a subset of \( \{a, b\} \), and so \( \{a\} \) is an element of the power set of \( A \).

6. (a) \( x \not\in A \cap B \) if and only if \( x \not\in A \) or \( x \not\in B \).

(b) \( x \not\in A \cup B \) if and only if \( x \not\in A \) and \( x \not\in B \).

(c) \( x \not\in A - B \) if and only if \( x \in A^c \) or \( x \in B \).

8. (a) \( A \cap B = \{7, 9, 11, 13, \ldots\} \).

(b) \( A \cup B = \{1, 3, 5, 7, 8, 9, 10, 11, \ldots\} \).

(c) \( (A \cup B)^c = \{2, 4, 6\} \).

(d) \( A^c \cap B^c = \{2, 4, 6\} \).

(e) \( (A \cup B) \cap C = \{3, 9, 12, 15, 18, 21, \ldots\} \).
(f) \((A \cap C) \cup (B \cap C) = \{3, 9, 12, 15, 18, 21, \ldots\}\).

(g) \(B \cap D = \{\}\).

(h) \((B \cap D)^c = \{1, 2, 3, \ldots\}\).

(i) \(A - D = \{7, 9, 11, 13, 15, \ldots\}\).

(j) \(B - D = \{1, 3, 5, 7, 9, \ldots\}\).

(k) \((A - D) \cup (B - D) = \{1, 3, 5, 7, 9, \ldots\}\).

(l) \((A \cup B) - D = \{1, 3, 5, 7, 9, \ldots\}\).

9. (a) For all \(x \in U\), if \(x \in P - Q\) then \(x \in R \cap S\).

(b) There is an \(x\) such that \(x \in P - Q\) and \(x \notin R \cap S\).

(c) For all \(x \in U\), if \(x \notin R \cap S\), then \(x \notin P - Q\).

11. Picture rectangles with lots of circles inside them!